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ON THE ANALYSIS AND SYNTHESIS OF CONTROL SYSTEMS USING A WORST CASE DISTURBANCE APPROACH

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ABSTRACT

Morse, Alfred Stephen. Ph.D., Purdue University, June 1967. On the Analysis and Synthesis of Control Systems Using a Worst Case Disturbance Approach. Major Professor: Violet B. Haas.

In this thesis, a study is made of two problems resulting from a "worst case disturbance" approach to the design of disturbed control systems. Using this approach, an error analysis problem is formulated. The problem is to determine the worst case (maximum) value of a system error index due to a bounded disturbance acting on a linear system. It it shown that this may be accomplished by computing the maximum value of a scalar function of known form with respect to a set of parameters which are restricted in magnitude. An efficient computational algorithm is developed which may be used in computing the maximum value of this function.

The formulation of a worst case disturbance min-max problem is presented. The problem is to determine a controller which will result in the smallest worst case value of a system error index. The relationship between this min-max problem and the differential game is indicated. Min-max problems for general linear systems and three different types of performance indices are investigated. It is shown that the min-max solutions of two of these problems do not always exist. In the third problem investigated, which involves a bounded control and a bounded disturbance, the complete min-max solution is found. Specific examples, demonstrating the results for all three problems, are presented.

CHAPTER 1

INTRODUCTION

1.1 Motivation

A basic engineering task of continuing importance is the problem of adequately controlling a system in the presence of an unknown external disturbance. In designing a controller to accomplish this task, the engineer must take into account not only the physical plant to be controlled, but also the external disturbance which may act on it. Since the actual disturbance is not known at the design stage, one must make some reasonable assumptions about the disturbance in order to carry out the design of the controller.

The Worst Case Disturbance Approach is an intuitively appealing way of dealing with this problem without requiring unreasonable or overly restrictive assumptions about the disturbance. In general terms, this approach may be outlined as follows. First it is assumed that the disturbance is suitably restrained (i.e., magnitude limited). Next, for a given controller, the worst possible (maximum) value of a prescribed system error index due to this disturbance is computed. This value is used to judge the quality of the system. If it is too large, the controller is altered and the technique reapplied. This process continues until a controller is found which will keep the corresponding worst case value of the system error index at an acceptably low level.

A controller determined in this manner is referred to as a "worst case disturbance" control.

There are two important research problems associated with the worst case disturbance approach. The first (the Worst Case Disturbance Analysis Problem) is the problem of actually determining the worst case value of a system error index when the system is under the influence of a given controller. The second (the Min-Max Problem) is the problem of synthesizing a controller which will result in the smallest worst case value of a given system error index. The research reported on the following pages has been directed towards the understanding of and possible solution to these two problems.

1.2 Alternate Approaches

There are several alternate approaches which one might consider in designing a control system in the presence of a disturbance. Probably the most straightforward is to assume that the disturbance is of a specific form, say a sinusoid or a step [1]. A controller can then be designed to make the value of a system performance index, which is determined using the assumed disturbance, as small as required. Unfortunately, information sufficient to enable the designer to make such an assumption about the disturbance is often unavailable, thus limiting the general use of this approach.

A second approach to the problem is to assume that the disturbance can be represented as a random variable [2,3]. In this instance, the control is designed to make some statistical average of the value of a system error index as small as required. Although the statistical approach often leads to systematic methods for dealing with an unknown disturbance, the approach seems to have several fundamental drawbacks. First, an adequate description of the disturbance, which is usually assumed to be available, is not always known. Even if some of the properties of the random process are known, they may be of a type not amenable to statistical methods (i.e., non-Gaussian, nonstationary random processes). A more basic criticism of the approach is that a statistical average of the value of a system error index is not a meaningful indicator of system performance. Whereas the use of such criteria is of recognized value in communication system design, their use in control system design is difficult to justify.

The most significant criticism of the worst case disturbance approach is that it may be overly conservative. Although this criticism is certainly valid in some instances, there are certainly problems in which a very conservative approach is justified; i.e., problems in which human life or very expensive equipment is at stake.

The most important advantage of the worst case disturbance approach is that it provides the designer with an upper bound for the value of a system error index which he knows will not be exceeded. Thus, if the controller is designed so that this upper bound is an acceptably low value, the designer knows that the system will operate properly in the presence of any disturbance. Hence one may place a much larger degree of confidence in a worst case disturbance controller than in a controller designed using either of the two previously discussed approaches.

1.3 Background

The formulation of the worst case disturbance approach for control systems is due to Howard [4,5]. In his work he formulated a general worst case error analysis problem and used variational techniques [6] and reachable zone theory [7] to study it. Saridis and Rekasius [8], also using a variational approach, studied an error analysis problem involving a bounded disturbance with a bounded rate of change. While the work reported by these researchers is applicable to a wide class of linear and nonlinear systems, their results in no way take into account the local or global nature of their solutions.†

Some work has been done on the actual design of control systems using a worst case disturbance approach. Jackson [9] has considered a design problem in which both the disturbance and its rate of change are bounded. His results are limited to second-order, linear, stationary systems. Graham [10] has used a related approach in the design of an attitude controller for a space booster. Koivuniemi [11] has also considered this approach, but failed to take into account the possibility of local solutions.

A formulation of the min-max problem studied in Chapter 3 can be found in [12]. Although there is almost no theoretical work reported in the literature on this problem, a closely related problem, the differential game, has received a considerable amount of attention. A recently published book by Rufas Isaacs [13] contains a number of specific examples of differential games. Work on the general theory of differential games has been reported in [14-16]. A study of some of the computational aspects of these problems is included in [16]. Other work dealing with this subject can be found in [17-20].

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†In any maximization problem, one may find several local maxima. The global maximum is the largest of these local maxima.

A partial differential equation similar to the Hamilton-Jacobi equation of variational calculus [21] is used in the study in Chapter 3. Though this equation can be found in most of the above references, it is derived heuristically using the Principle of Optimality [12]. The simple derivation provides insight into the nature of the min-max problem. The derivation appears in Appendix B.

1.4 Organization

The research reported on the following pages is divided into two main chapters. The first deals with the worst case disturbance analysis problem. In studying this problem, emphasis is placed on developing a sound method for computing the worst case value of a system error index.

Chapter 3 contains the results of an investigation of min-max problems for a general class of linear systems. Problems involving three different types of performances are analyzed and compared. In the interest of continuity, the proofs of the results of this chapter are given in a separate appendix.

Finally, in Chapter 4, the material in this report is briefly summarized. Several suggestions for future study are indicated. The important contributions of this research are also cited in this chapter.

1.5 Assumptions

In the work that follows, several basic assumptions are made. The first is that the plant under investigation may be described by a linear differential equation. This equation may be of any order and time-varying.

Unless otherwise stated, it is assumed that the magnitude of the disturbance is bounded, and that the bounds are known. Without this assumption, in many problems the worst case disturbance would be infinite, leading to meaningless results. Furthermore, real disturbances are indeed bounded, and one is often in a position to make reasonable estimates of these bounds.

Performance indices of several different forms are considered. It is assumed that the particular weightings of the terms in these performance indices are given. In practice, the weightings of these terms would be determined from general system requirements in such a way that if the system performance index is small, the requirements of the system are being attained. The determination of proper weightings will usually involve some trial and error.

In the min-max problems studied in Chapter 3, it is assumed that the disturbance may not be measured directly. In practice, a signal representing the measured value of a disturbance must come from the output of a sensor describable by a differential equation. Thus this signal may be considered as a state variable of the system.

For obvious reasons, only feedback control is considered in this work. Without knowing a disturbance in advance, it is impossible to control a disturbed system without feedback. In the worst case disturbance analysis problem studied in Chapter 2, it is presumed that a linear feedback control is included in the description of the system.

A point of view taken in all of the following work is that <u>only</u> the globally worst case value of a system error index or performance index has meaning. There are two reasons for this. First, if one accepts a local maximum as the worst case value of the error index and chooses a controller to make this value small, he may in reality be increasing the globally worst case value of the error index. Second, without knowledge of the global maximum, the designer has no way of knowing if the system will operate acceptably under worst case conditions.

1.6 Notation and Terminology

Throughout this work, state variable notation is used. Vectors and matrices are underlined, scalars are not. Unless otherwise stated, all vectors are column vectors. A vector or matrix is called continuous or differentiable if all of the elements of the vector or matrix are continuous or differentiable. The symbol $\dot{\underline{x}}(t)$ represents a vector whose elements are the time derivatives of the elements of $\underline{x}(t)$. A prime (′) is used to indicate the transpose of a matrix. The inner product of two n-vectors $\underline{x}$ and $\underline{y}$ is represented by

$$\underline{x}'\underline{y} = \sum_{i=1}^{n} x_i \, y_i \tag{1.1}$$

where x_i and y_i are elements of the vectors. The norm of an n-vector $\underline{x}$ is defined as

$$||\underline{x}||_{\underline{Q}} = [\underline{x}'\underline{Q}\,\underline{x}]^{\frac{1}{2}} \tag{1.2}$$

where $\underline{Q}$ is a positive-semidefinite nxn matrix. The symbol $\underline{F}_x(\underline{x})$ is used to represent an n-vector whose i^{th} component is

$$\frac{dF(\underline{x})}{dx_i} ,$$

$F(\underline{x})$ being a scalar function of $\underline{x}$.

A scalar function $F(\underline{x})$ is called multimodal if it has more than one local maximum [22]. A set X is called convex if for every pair of points $\underline{x}$, $\underline{y}$ in X, the point

$$\underline{z} = (1 - \lambda)\, \underline{x} + \lambda\, \underline{y} \;; \qquad 0 \leq \lambda \leq 1 \tag{1.3}$$

also is in X. A <u>function</u> $F(\underline{x})$ defined on a convex set X is called <u>convex</u> if for every pair of points $\underline{x}$ and $\underline{y}$ in X

$$F((1 - \lambda)\, \underline{x} + \lambda\, \underline{y}) \leq (1 - \lambda)\, F(\underline{x}) + \lambda\, F(\underline{y}) \tag{1.4}$$

for $0 \leq \lambda \leq 1$. In Chapter 2, use is made of the following property of convex functions with continuous derivatives [23]:

$$F(\underline{x}) \geq F(\underline{y}) + \underline{F}'_{\underline{x}}(\underline{y})\ (\underline{x} - \underline{y}) \tag{1.5}$$

Finally, it is convenient to define the function

$$\mathrm{sgn}(q,y) = \begin{cases} 1 & \text{if } y > 0 \\ q & \text{if } y = 0 \\ -1 & \text{if } y < 0 \end{cases} \tag{1.6}$$

where q and y are scalars.

CHAPTER 2

A WORST CASE DISTURBANCE ANALYSIS PROBLEM

2.1 Introduction

In this chapter a worst case disturbance analysis problem for linear systems is formulated. The first logical step towards the solution of this problem is to invoke the necessary conditions of the Maximum Principle [6]. Unfortunately, these conditions lead to a rather difficult two-point boundary value problem. It will be shown in the work that follows that this boundary value problem may be avoided by introducing a special function which is related to the worst case value of the system error index. Through a discussion of the analytic and geometric properties of this function, an understanding of the underlying nature of the error analysis problem can be obtained.

2.2 General Problem Statement

It is assumed that the system to be analyzed can be described by the linear vector differential equation

$$\dot{\underline{x}}(t) = \underline{A}(t)\ \underline{x}(t) + \underline{c}(t)\ v(t) \tag{2.1}$$

where $\underline{x}(t)$ is an n-vector describing the state of the system, $\underline{A}(t)$ and $\underline{c}(t)$ are time-varying matrices, and $v(t)$ is a scalar forcing function representing a disturbance acting on the system. It is assumed that the elements of $\underline{A}(t)$ and $\underline{c}(t)$ are piecewise-continuous on $[t_o, T]$ where t_o

is a fixed initial time and T a fixed terminal time. The function $v(t)$ is assumed to be a member of the set S_v. S_v is the set of piecewise-continuous time functions defined on $[t_o,T]$ and satisfying

$$\left.\begin{array}{l}\gamma_1(t) \leq v(t) \leq \gamma_2(t) \\ \gamma_1(t) < \gamma_2(t)\end{array}\right\} \quad t \in [t_o,T] \tag{2.2}$$

where $\gamma_1(t)$ and $\gamma_2(t)$ are piecewise-continuous on $[t_o,T]$. The initial state

$$\underline{x}(t_o) = \underline{x}_o \tag{2.3}$$

is assumed to be known.

Next, define the error function

$$E(t) = ||\underline{x}(t)||_{\underline{Q}} \tag{2.4}$$

where $\underline{Q}$ is a positive-semidefinite constant matrix. Finally, to judge the quality of the system we will define a performance index simply as

$$J(\underline{x}_o,\ t_o,\ T;\ v) = \frac{1}{2}\,E^2(T) \tag{2.5}$$

The worst case disturbance problem may now be stated:

Determine the disturbance $v^*(t) \in S_v$ such that

$$J(\underline{x}_o,\ t_o,\ T;\ v^*) \geq J(\underline{x}_o,\ t_o,\ T;\ v) \tag{2.6}$$

for all $v(t) \in S_v$.

2.3 A Special Case

Consider first the special case of (2.4) in which

$$E(t) = |\underline{K}'\underline{x}(t)| \tag{2.7}$$

where K is a constant n-vector. Since the disturbance which will maximize $J(\underline{x}_o,\ t_o,\ T\ ;\ v)$ will also maximize $E(T)$, we focus attention on the error function.

From linear differential equation theory [24] it is well known that $\underline{x}(T)$ may be expressed as

$$\underline{x}(T) = \underline{\psi}'(t)\ \underline{x}(t) + \int_t^T \underline{\psi}'(\tau)\ \underline{c}(\tau)\ v(\tau)\ d\tau \qquad t \in [t_o, T] \tag{2.8}$$

where $\underline{\psi}(t)$ is an nxn matrix satisfying

$$\dot{\underline{\psi}}(t) = -\ \underline{A}'(t)\ \underline{\psi}(t) \qquad t \in [t_o, T] \tag{2.9}$$

with terminal conditions

$$\underline{\psi}(T) = \underline{I}_n \qquad \text{(nxn identity matrix)} \tag{2.10}$$

The terminal error may now be expressed as

$$E(T) = \left| \underline{K}'\underline{\psi}'(t_o)\ \underline{x}(t_o) + \int_{t_o}^T \underline{K}'\underline{\psi}'(\tau)\ \underline{c}(\tau)\ v(\tau)\ d\tau \right| \tag{2.11}$$

It will be convenient to let

$$v(t) = \frac{\gamma_2(t) + \gamma_1(t)}{2} + \left(\frac{\gamma_2(t) - \gamma_1(t)}{2} \right) \rho(t) \tag{2.12}$$

where $\rho(t)$ is a piecewise-continuous scalar time function satisfying

$$|\rho(t)| \leq 1 \qquad t \in [t_o, T] \tag{2.13}$$

Because of the obvious one to one relationship between $v(t)$ and $\rho(t)$, it will be sufficient to maximize $E(T)$ with respect to $\rho(t)$. The error function in equation (2.11) may now be written as

$$E(T) = \Big| \underline{K}'\underline{\psi}'(t_o)\ \underline{x}(t_o) + \theta(t_o)$$

$$+ \int_{t_o}^T \underline{K}'\underline{\psi}'(\tau)\ \underline{c}(\tau)\ \frac{(\gamma_2(\tau) - \gamma_1(\tau))}{2}\ \rho(\tau)\ d\tau \Big| \tag{2.14}$$

where

$$\theta(t_o) = \int_{t_o}^T \frac{(\gamma_2(\tau) + \gamma_1(\tau))}{2}\ \underline{K}'\underline{\psi}'(\tau)\ \underline{c}(\tau)\ d\tau \tag{2.15}$$

It is clear by inspection of (2.14) that if

$$\underline{K}'\underline{\psi}'(t_o)\ \underline{x}(t_o) + \theta(t_o) \neq 0$$

the maximizing $\rho(t)$ is

$$\rho^*(t) = \left\{ \begin{array}{lll} 1 & \text{if} & (\underline{c}'(t)\ \underline{\psi}(t)\ \underline{K})(\underline{K}'\underline{\psi}'(t_o)\ \underline{x}(t_o) + \theta(t_o)) > 0 \\ -1 & \text{if} & (\underline{c}'(t)\ \underline{\psi}(t)\ \underline{K})(\underline{K}'\underline{\psi}'(t_o)\ \underline{x}(t_o) + \theta(t_o)) < 0 \end{array} \right\} \tag{2.16}$$

If $\underline{c}'(t)\ \ \underline{\psi}(t)\ \underline{K} = 0$ on some subinterval of $[t_o,T]$, it is clear that $\rho(t)$ has no effect on E(T) on this subinterval; it may therefore be chosen to be any value. To be definite we shall set $\rho^*(t) = 0$ on any such subinterval.

Inspection of (2.14) further indicates that if

$$\underline{K}'\underline{\psi}'(t_o)\ \underline{x}(t_o) + \theta(t_o) = 0$$

the maximizing $\rho(t)$ is not unique. That is, if

$$\rho^*(t) = \left\{ \begin{array}{lll} 1 & \text{if} & \underline{c}'(t)\ \underline{\psi}(t)\ \underline{K} > 0 \\ -1 & \text{if} & \underline{c}'(t)\ \underline{\psi}(t)\ \underline{K} < 0 \end{array} \right\} \tag{2.17}$$

it is clear that $-\ \rho^*(t)$ will also maximize E(T). Since we are primarily concerned with the worst case value of E(T) which is the same for either choice of $\rho(t)$, we shall find it convenient to define a specific maximizing disturbance. Thus, if

$$\underline{K}'\underline{\psi}'(t_o)\ \underline{x}(t_o) + \theta(t_o) = 0$$

it will be assumed that $\rho^*(t)$ is as defined in (2.17). All of the above statements lead to the following expression for $v^*(t)$:

$$v^*(t) = \frac{[\gamma_2(t) - \gamma_1(t)]}{2}[\text{sgn}(0,\ \underline{c}'(t)\ \underline{\psi}(t)\ \underline{K})]$$
$$[\text{sgn}(1,\ \underline{K}'\underline{\psi}'(t_o)\ \underline{x}(t_o) + \theta(t_o))] + \frac{[\gamma_2(t) + \gamma_1(t)]}{2} \tag{2.18}$$

By invoking the Principle of Optimality [12], one may replace the argument t_o with t in the above expression without affecting $v^*(t)$. The resulting expression for $v^*(t)$ is therefore a function of the present state or a "feedback" disturbance.

The worst case disturbance value of the terminal error may now be expressed as

$$E^*(T) = |\underline{K}'\underline{\psi}'(t_o)\ \underline{x}(t_o) + \theta(t_o)| + \int_{t_o}^{T} \frac{[\gamma_2(\tau) - \gamma_1(\tau)]}{2} |\underline{c}'(\tau)\ \underline{\psi}(\tau)\ \underline{K}|\ d\tau \tag{2.19}$$

Again using the reasoning of the Principle of Optimality, t_o may be replaced by $t \in [t_o, T]$ without affecting the value of the above expression.

Though independently developed by this author, the above results are not new. Jackson [9] in 1960, Desoer [25] in 1962, and Howard [4] in 1964 have all obtained similar solutions.

The above results are illustrated by the following example:

Example 2.3.1

Consider the oscillator

$$\ddot{y} + y = v(t)$$

where

$$|v(t)| \leq a$$

Define

$$E(T) = |y(T)|$$

The problem is to determine the maximum value of E(T). If $x_1 = y$ and $x_2 = \dot{y}$, the $\underline{\psi}$ matrix may be expressed as

$$\underline{\psi}(t) = \begin{pmatrix} \cos(T-t) & \sin(T-t) \\ -\sin(T-t) & \cos(T-t) \end{pmatrix}$$

Thus

$$v^*(t) = -a\,[\mathrm{sgn}(0, \sin(T-t))]$$
$$[\mathrm{sgn}(1, (x_1(t)\cos(T-t) - x_2(t)\sin(T-t))]$$

Using (2.19), the worst case disturbance value of E(T) is computed to be

$$E^*(T) = |\cos(T-t)\,x_1(t) - \sin(T-t)\,x_2(t)|$$
$$+ a\,[2\ell + 1 - |\cos(T-t)|]$$

where $$\ell \leq \frac{T-t}{\pi} \leq \ell + 1$$

and ℓ is any non-negative integer.

2.4 The Maximum Principle Equations for the General Problem

We now return to the general problem of Section 2.2. Since the maximization of $J(\underline{x}_o, t_o, T; v)$ with respect to v is a problem in the calculus of variations, the Maximum Principle [6] is employed as a first step towards solution.

Define the Hamiltonian

$$H(\underline{x}(t), \underline{\lambda}(t), v(t), t) = \underline{\lambda}'(t)\,\underline{A}(t)\,\underline{x}(t) + \underline{\lambda}'(t)\,\underline{c}(t)\,v(t) \qquad (2.20)$$

If $v^*(t)$ is a maximizing disturbance resulting in a trajectory $\underline{x}^*(t)$ $t \in [t_o,T]$, then the following necessary conditions must hold:

1. There exist multipliers λ_o and $\underline{\lambda}(t)$ which are continuous on $[t_o,T]$ and which do not all vanish simultaneously on $[t_o,T]$.

2. The trajectory $\underline{x}^*(t)$ and the disturbance $v^*(t)$ are related on $[t_o,T]$ by

$$\left.\begin{aligned} &\dot{\underline{x}}^*(t) = \underline{A}(t)\,\underline{x}^*(t) + \underline{c}(t)\,v^*(t) \\ &\text{with} \qquad \underline{x}^*(t_o) = \underline{x}_o \end{aligned}\right\} \qquad (2.21)$$

3. The multipliers λ_o and $\underline{\lambda}(t)$ satisfy the differential equations

$$\left.\begin{aligned} \dot{\lambda}_o &= 0 \\ \dot{\underline{\lambda}}(t) &= -\,\underline{A}'(t)\,\underline{\lambda}(t) \end{aligned}\right\} \tag{2.22}$$

on $[t_o, T]$.

4. At the terminal time

$$\underline{\lambda}(T) = \lambda_o \underline{Q}\underline{x}^*(T) \tag{2.23}$$

5. For all $t \in [t_o, T]$ and $v(t) \in S_v$

$$H(\underline{x}^*(t), \underline{\lambda}(t), v^*(t), t) \geq H(\underline{x}^*(t), \underline{\lambda}(t), v(t), t) \tag{2.24}$$

Note that from condition 3, λ_o must be a constant. From conditions 3 and 4 it is clear that if $\lambda_o = 0$, $\underline{\lambda}(t) \equiv 0$. This would violate condition 1. Thus $\lambda_o \neq 0$. We choose it to be unity.

Note that condition 5 implies that

$$v^*(t) = \frac{[\gamma_2(t) - \gamma_1(t)]}{2} \operatorname{sgn}(0, \underline{c}'(t)\,\underline{\lambda}(t)) + \frac{\gamma_2(t) + \gamma_1(t)}{2} \tag{2.25}$$

at all points on $[t_o, T]$ where $\underline{c}'(t)\,\underline{\lambda}(t) \neq 0$. If $\underline{c}'(t)\,\underline{\lambda}(t) \equiv 0$ on some subinterval of $[t_o, T]$, the trajectory $\underline{x}^*(t)$ is said to be on a singular surface [26]. On a singular surface, condition 5 of the Maximum Principle is satisfied for any $v(t)$. The problem of determining $v^*(t)$ on a singular surface will be solved in the next section.

2.5 A Simplification of the General Problem

The necessary conditions outlined in the last section imply that one must solve a two-point boundary value problem in order to find the maximum value of $J(\underline{x}_o, t_o, T; v)$. By introducing a special function, we shall show that this boundary value problem may be avoided. The idea is basically as follows. In Section 2.3 a complete solution was obtained for the special problem with the error function defined by equation (2.7). Thus, associated with the special problem there must be multipliers which satisfy the differential equations (2.22) and boundary conditions (2.23) of the Maximum Principle. It is noted that the multipliers for the general problem of Section 2.2 also satisfy these same differential equations but with different boundary conditions. If an error function can somehow be judiciously defined for the special problem so that the multipliers for both problems satisfy the same boundary conditions, then the general problem will be solved.

Consider the error function

$$E(t,\underline{\alpha}) = |\underline{x}'(t)\, \underline{M}^{-1}\, \underline{\alpha}| \tag{2.26}$$

where $\underline{\alpha} \in E^n$ is an arbitrary constant n-vector. $\underline{M}$ is a nonsingular matrix which diagonalizes $\underline{Q}$:

$$\underline{M}\,\underline{Q}\,\underline{M}' = \left(\begin{array}{c|c} \underline{I}_r & \underline{0} \\ \hline \underline{0} & \underline{0} \end{array}\right) \triangleq \underline{D} \tag{2.27}$$

$\underline{I}_r$ is the rxr identity matrix, r being the rank of $\underline{Q}$. The other entries in the partitioned matrix in (2.27) are null matrices of appropriate dimension.

Consider the performance index

$$\bar{J}(\underline{x}_o,\ t_o,\ T,\ \underline{\alpha};\ v) = \frac{E^2(T,\underline{\alpha})}{2} \tag{2.28}$$

The worst case disturbance value of this performance index can be related to the worst case disturbance value of the general performance index $J(\underline{x}_o,\ t_o,\ T;\ v)$ defined by equations (2.4) and (2.5).

Define the return $F(\underline{x}(t),\ t,\ \underline{\alpha})$ as the worst case value of $E^2(T,\underline{\alpha})/2$ for a process starting at time $t \in [t_o,T]$ in state $\underline{x}(t)$.

$$F(\underline{x}(t),\ t,\ \underline{\alpha}) = \max_{\substack{v(\tau)\epsilon S_v \\ \tau\epsilon[t,T]}} \left[\frac{E^2(T,\underline{\alpha})}{2} \right] \tag{2.29}$$

From the results in Section 2.3 it may be readily determined that

$$F(\underline{x}(t),\ t,\ \underline{\alpha}) = \frac{1}{2} [\,|\underline{x}'(t)\ \underline{P}(t)\ \underline{\alpha} + \theta(t,\underline{\alpha})| + \phi(t,\underline{\alpha})]^2 \tag{2.30}$$

where

$$\dot{\underline{P}}(t) = -\ \underline{A}'(t)\ \underline{P}(t) \tag{2.31}$$

and

$$\underline{P}(T) = \underline{M}^{-1} \tag{2.32}$$

(We have introduced $\underline{P}(t) = \underline{M}^{-1}\ \underline{\psi}(t)$ to simplify notation.) The scalars $\phi(t,\underline{\alpha})$ and $\theta(t,\underline{\alpha})$ satisfy

$$\left.\begin{aligned} \phi(t,\underline{\alpha}) &= -\ |\underline{c}'(t)\ \underline{P}(t)\ \underline{\alpha}| \left[\frac{\gamma_2(t) - \gamma_1(t)}{2} \right] \\ \theta(t,\underline{\alpha}) &= -\ [\underline{c}'(t)\ \underline{P}(t)\ \underline{\alpha}] \left[\frac{\gamma_2(t) + \gamma_1(t)}{2} \right] \end{aligned}\right\} \tag{2.33}$$

with boundary conditions

$$\left.\begin{aligned} \phi(T,\underline{\alpha}) &= 0 \\ \theta(T,\underline{\alpha}) &= 0 \end{aligned}\right\} \tag{2.34}$$

Define the vector function $\underline{\lambda}(t,\underline{\alpha})$ on $[t_o,T]$ by

$$\begin{aligned} \underline{\lambda}(t,\underline{\alpha}) = \underline{P}(t)\ \underline{\alpha}\ [\mathrm{sgn}(1,\ (\underline{x}'(t,\underline{\alpha})\ \underline{P}(t)\ \underline{\alpha} + \theta(t,\underline{\alpha})))] \\ [\,|\underline{x}'(t,\underline{\alpha})\ \underline{P}(t)\ \underline{\alpha} + \theta(t,\underline{\alpha})| + \phi(t,\underline{\alpha})] \end{aligned} \tag{2.35}$$

where $\underline{x}(t,\underline{\alpha})$ is the worst case trajectory for the performance index defined in (2.28). It may be easily determined from the results of Section 2.3 that the worst case disturbance for $\bar{J}$ is

$$\begin{aligned} v(t,\underline{\alpha}) = \left[\frac{\gamma_2(t) - \gamma_1(t)}{2} \right] \mathrm{sgn}(0,\ \underline{c}'(t)\ \underline{\lambda}(t,\underline{\alpha})) \\ + \left[\frac{\gamma_2(t) + \gamma_1(t)}{2} \right] \end{aligned} \tag{2.36}$$

Next define Ω^r as a subspace of E^n with the following properties: A vector $\underline{\alpha}$ is in Ω^r if

$$\left.\begin{array}{ll} 1) & \underline{\alpha}'\underline{\alpha} = 1 \\ 2) & \alpha_i = 0 \qquad i = r + 1 \text{ to } n \end{array}\right\} \tag{2.37}$$

Next consider the vector $\underline{F}_\alpha(\underline{x}_o, t_o, \underline{\alpha})$ whose i^{th} component is

$$\frac{dF}{d\alpha_i}(\underline{x}_o, t_o, \underline{\alpha})$$

Inspection of (2.30) indicates that these derivatives will exist for all $\underline{\alpha} \in E^n$ if

$$\frac{d|\xi|}{d\,\xi} \triangleq \mathrm{sgn}(1,\xi)$$

With these preliminaries completed, the following theorem may be stated:

<u>Theorem 2.1</u>

If there exists an $\hat{\underline{\alpha}} \in \Omega^r$ such that

$$\underline{D}\,\underline{F}_\alpha(\underline{x}_o, t_o, \hat{\underline{\alpha}}) = 2(F(\underline{x}_o, t_o, \hat{\underline{\alpha}}))\,\hat{\underline{\alpha}} \tag{2.38}$$

then the trajectory $\underline{x}(t,\hat{\underline{\alpha}})$, $\underline{\lambda}(t,\hat{\underline{\alpha}})$ and the disturbance $v(t,\hat{\underline{\alpha}})$ constitute a solution of the necessary conditions of the Maximum Principle for the <u>general</u> problem formulated in Section 2.2.

In Appendix A it is shown that $\underline{\lambda}(t,\hat{\underline{\alpha}})$ is continuous and satisfies the differential equation in (2.22) on $[t_o,T]$, thus establishing that necessary conditions 1 and 3 are satisfied. Condition 2 is automatically satisfied since $\underline{x}(t,\hat{\underline{\alpha}})$ is the trajectory resulting from the disturbance $v(t,\hat{\underline{\alpha}})$. Next, from (2.25) and (2.36) it is clear that necessary condition 5 is satisfied. Thus to prove Theorem 2.1 it remains to be shown that if the hypothesis holds, necessary condition 4 (the transversality condition) holds. To do this first note from (2.30) that

$$\underline{F}_\alpha(\underline{x}_o, t_o, \hat{\underline{\alpha}}) = \Big\{ \Big[\underline{P}'(t_o)\, \underline{x}(t_o) + \int_{t_o}^{T} \Big(\frac{\gamma_2(\tau) + \gamma_1(\tau)}{2}\Big) \underline{P}'(\tau)\, \underline{c}(\tau)\, d\tau\Big]$$

$$\mathrm{sgn}(1, (\underline{x}_o'\, \underline{P}(t_o)\, \hat{\underline{\alpha}} + \theta(t_o,\hat{\underline{\alpha}}))$$

$$+ \int_{t_o}^{T} \Big(\frac{\gamma_2(\tau) - \gamma_1(\tau)}{2}\Big) \underline{P}'(\tau)\, \underline{c}(\tau)\ \mathrm{sgn}(0, \underline{c}'(\tau)\, \underline{P}(\tau)\, \hat{\underline{\alpha}}) d\tau \Big\} E(T,\hat{\underline{\alpha}}) \tag{2.39}$$

But from (2.8), (2.32) to (2.34) and (2.36) it follows that

$$\underline{F}_\alpha(\underline{x}_o, t_o, \hat{\underline{\alpha}}) = (\underline{M}^{-1})'\, \underline{x}(T,\hat{\underline{\alpha}})\, E(T,\hat{\underline{\alpha}})\ \mathrm{sgn}(1, (\underline{x}_o'\, \underline{P}(t_o)\, \hat{\underline{\alpha}} + \theta(t_o,\hat{\underline{\alpha}}))) \tag{2.40}$$

From the hypothesis of the theorem

$$2\, F(\underline{x}_o, t_o, \hat{\underline{\alpha}})\, \hat{\underline{\alpha}} = E^2(T,\hat{\underline{\alpha}})\, \hat{\underline{\alpha}} = \underline{D}\, \underline{F}_\alpha(\underline{x}_o, t_o, \hat{\underline{\alpha}}) \tag{2.41}$$

Thus

$$E^2(T,\hat{\underline{\alpha}})\, \hat{\underline{\alpha}} = \underline{D}(\underline{M}^{-1})'\, \underline{x}(T,\hat{\underline{\alpha}})\, E(T,\hat{\underline{\alpha}})\ \mathrm{sgn}(1, (\underline{x}_o'\, \underline{P}(t_o)\, \hat{\underline{\alpha}} + \theta(t_o,\hat{\underline{\alpha}}))) \tag{2.42}$$

or

$$\hat{\underline{\alpha}} = \frac{\underline{D}(\underline{M}^{-1})'\, \underline{x}(T,\hat{\underline{\alpha}})\ \mathrm{sgn}(1, (\underline{x}_o'\, \underline{P}(t_o)\, \hat{\underline{\alpha}} + \theta(t_o,\hat{\underline{\alpha}})))}{E(T,\hat{\underline{\alpha}})} \tag{2.43}$$

From (2.35)

$$\underline{\lambda}(T,\hat{\underline{\alpha}}) = \underline{M}^{-1}\, \hat{\underline{\alpha}}\, [\,\mathrm{sgn}(1, (\underline{x}'(T,\hat{\underline{\alpha}})\, \underline{P}(T)\, \hat{\underline{\alpha}} + \theta(T,\hat{\underline{\alpha}})))]\ E(T,\hat{\underline{\alpha}}) \tag{2.44}$$

As a result of the proofs given in Appendix A

$$\mathrm{sgn}(1, (\underline{x}_o\, \underline{P}(t_o)\, \hat{\underline{\alpha}} + \theta(t_o,\hat{\underline{\alpha}}))) = \mathrm{sgn}(1, \underline{x}(T,\underline{\alpha})\, \underline{P}(T)\, \hat{\underline{\alpha}} + \theta(T,\hat{\underline{\alpha}})) \tag{2.45}$$

Thus using (2.43) and (2.44)

$$\underline{\lambda}(T,\hat{\underline{\alpha}}) = \underline{M}^{-1}\, \underline{D}(\underline{M}^{-1})'\, \underline{x}(T,\hat{\underline{\alpha}}) = \underline{Q}\, \underline{x}(T,\hat{\underline{\alpha}}) \tag{2.46}$$

Thus the transversality conditions is satisfied and the theorem is proved.

The converse of Theorem 2.1 is also true:

Theorem 2.2

If for some $\underline{\hat{\alpha}} \in \Omega^r$, the trajectory $\underline{x}(t,\underline{\hat{\alpha}})$, $\underline{\lambda}(t,\underline{\hat{\alpha}})$ resulting from the disturbance $v(t,\underline{\alpha})$ satisfies the necessary conditions of the Maximum Principle for the general problem, then

$$\underline{D}\ \underline{F}_\alpha(\underline{x}_o,\ t_o,\ \underline{\hat{\alpha}}) = 2\ [F(\underline{x}_o,\ t_o,\ \underline{\hat{\alpha}})]\ \underline{\hat{\alpha}} \tag{2.47}$$

Since the necessary conditions are satisfied

$$\underline{\lambda}(T,\underline{\hat{\alpha}}) = \underline{Q}\ \underline{x}(T,\underline{\hat{\alpha}}) \tag{2.48}$$

From (2.35) and (2.48)

$$\underline{Q}\ \underline{x}(T,\underline{\hat{\alpha}}) = \underline{M}^{-1}\ \underline{\hat{\alpha}}\ E(T,\underline{\hat{\alpha}})\ \mathrm{sgn}(1,\ \underline{x}'(T)\ \underline{P}(T)\ \underline{\hat{\alpha}} + \theta(T,\underline{\alpha})) \tag{2.49}$$

Premultiplying by $\underline{M}$ and using (2.27)

$$\underline{D}(\underline{M}^{-1})'\ \underline{x}(T,\underline{\hat{\alpha}}) = \underline{\hat{\alpha}}\ E(T,\underline{\hat{\alpha}})\ \mathrm{sgn}(1,\ \underline{x}'(T)\ \underline{\hat{\alpha}} + \theta(T,\underline{\alpha})) \tag{2.50}$$

or

$$\begin{aligned} &\underline{D}(\underline{M}^{-1})'\ \underline{x}(T,\underline{\hat{\alpha}})\ E(T,\underline{\hat{\alpha}})\ \mathrm{sgn}(1,\ \underline{x}(T)\ \underline{P}(T)\ \underline{\hat{\alpha}} + \theta(T,\underline{\hat{\alpha}})) \\ &= \underline{\hat{\alpha}}\ E^2(T,\underline{\hat{\alpha}}) = 2\ F(x_o,\ t_o,\ \underline{\hat{\alpha}})\ \underline{\hat{\alpha}} \end{aligned} \tag{2.51}$$

Finally, using (2.40), (2.45) and (2.51) it follows that

$$\underline{D}\ \underline{F}_\alpha(\underline{x}_o,\ t_o,\ \underline{\hat{\alpha}}) = 2\ F(\underline{x}_o,\ t_o,\ \underline{\hat{\alpha}})\ \underline{\hat{\alpha}} \tag{2.52}$$

and thus the theorem is proved.

There remains one important question. Does there exist an $\underline{\alpha} \in \Omega^r$ such that $v(t,\underline{\alpha})$ is indeed the maximizing disturbance for the general problem? This question is answered by a corollary of the following theorem.

Theorem 2.3

Let $J(\underline{x}_o,\ t_o,\ T;\ v)$ and $F(\underline{x}(t),\ t,\ \underline{\alpha})$ be defined by (2.5) and (2.29) respectively. Then

$$\max_{v \in S_v} J(\underline{x}_o, t_o, T; v) = \max_{\underline{\alpha} \in \Omega^r} F(\underline{x}_o, t_o, \underline{\alpha}) \tag{2.53}$$

To prove this result, first consider the difference between the performance indices for the general and special problems:

$$\begin{aligned} &J(\underline{x}_o, t_o, T; v) - \bar{J}(\underline{x}_o, t_o, T, \underline{\alpha}; v) \\ &= \frac{1}{2} [\underline{x}'(T)\, \underline{Q}\, \underline{x}(T) - (\underline{x}'(T)\, \underline{M}^{-1}\, \underline{\alpha})^2] \\ &= \frac{1}{2} ||\underline{D}(\underline{M}^{-1})'\, \underline{x}(T) - \underline{\alpha}(\underline{x}'(T)\, \underline{M}^{-1}\, \underline{\alpha})||^2 \end{aligned} \tag{2.54}$$

It is clear that

$$J(\underline{x}_o, t_o, T; v) \geq \bar{J}(\underline{x}_o, t_o, T, \underline{\alpha}; v) \tag{2.55}$$

for all $v \in S_v$ and $\underline{\alpha} \in \Omega^r$. In particular, if $v^*(t)$ is the maximizing disturbance for J,

$$J(\underline{x}_o, t_o, T; v^*) \geq J(\underline{x}_o, t_o, T; v) \geq \bar{J}(\underline{x}_o, t_o, T, \underline{\alpha}; v) \tag{2.56}$$

for all $\underline{\alpha} \in \Omega^r$, $v \in S_v$. Therefore

$$J(\underline{x}_o, t_o, T; v^*) \geq \bar{J}(\underline{x}_o, t_o, T; \underline{\alpha}, v(t,\underline{\alpha})) = F(\underline{x}_o, t_o, \underline{\alpha}) \tag{2.57}$$

for all $\underline{\alpha} \in \Omega^r$. If $\underline{x}^*(T)$ is the final state resulting from the disturbance $v^*(t)$, the vector $\underline{\alpha}^*$ may be chosen as

$$\underline{\alpha}^* = \frac{\underline{D}(\underline{M}^{-1})'\, \underline{x}^*(T)}{||\underline{D}(\underline{M}^{-1})'\, \underline{x}^*(T)||} \tag{2.58}$$

From (2.54) it is clear that

$$J(\underline{x}_o, t_o, T; v^*) = \bar{J}(\underline{x}_o, t_o, T, \underline{\alpha}^*; v^*) \tag{2.59}$$

Therefore

$$\begin{aligned} J(\underline{x}_o, t_o, T; v^*) &\geq \max_{\underline{\alpha} \in \Omega^r} F(\underline{x}_o, t_o, \underline{\alpha}) \geq F(\underline{x}_o, t_o, \underline{\alpha}^*) \\ &\geq \bar{J}(\underline{x}_o, t_o, T, \alpha^*; v^*) = J(\underline{x}_o, t_o, T; v^*) \end{aligned} \tag{2.60}$$

Thus the theorem is proved.

Corollary 2.1

If $v^*(t)$, $\underline{\alpha}^*$ and $v(t,\underline{\alpha})$ are as defined above, then

$$v^*(t) = v(t,\underline{\alpha}^*) \tag{2.61}$$

is a maximizing disturbance for the general problem.

This follows directly from Theorem 2.3.

Note that since $v(t,\underline{\alpha}^*)$ is not necessarily unique, $v^*(t)$ is not necessarily unique. However, if several different disturbances result in the worst case value of the performance index, this will cause no problem since it is the worst case value which is of primary concern.

In the last section, the possibility of a singular surface (i.e., $\underline{\lambda}'(t)\ \underline{c}(t) \equiv 0$ on an interval of finite length) was mentioned. From the discussion in Section 2.3 and the results stated above, it is clear that the disturbance $v^*(t)$ does not affect the performance index on a singular surface. By defining $v(t,\underline{\alpha})$ as in (2.36), $v^*(t)$ is specified to be zero on a singular surface.

Note that as a consequence of Theorem 2.3, equation (2.47) becomes a necessary condition to be satisfied by a maximizing $\underline{\alpha}$. Unfortunately this condition may be satisfied by more than one value of $\underline{\alpha}$; $F(x_o, t_o, \underline{\alpha})$ is a multimodal function (see Section 1.6) of $\underline{\alpha}$. More will be said about this in a later section.

2.6 Properties of $F(\underline{x}(t), t, \underline{\alpha})$

In this section, the previously derived results are summarized. In addition, several additional properties of $F(\underline{x}(t), t, \underline{\alpha})$ are described.

1. $F(\underline{x}(t), t, \underline{\alpha})$ is the optimal return for the performance index $\bar{J}(\underline{x}_o, t_o, T, \underline{\alpha}; v) = \frac{1}{2}\ (\underline{x}'(T)\ \underline{M}^{-1}\ \underline{\alpha})^2$

2. The trajectory $\underline{x}(t,\underline{\alpha})$, $\underline{\lambda}(t,\underline{\alpha})$ and the disturbance $v(t,\underline{\alpha})$ represent the optimal solution for the performance index $\bar{J}(\underline{x}_o, t_o, T, \underline{\alpha}; v)$.

3. The trajectory $\underline{x}(t,\underline{\alpha})$, $\underline{\lambda}(t,\underline{\alpha})$ and the disturbance $v(t,\underline{\alpha})$ satisfy the necessary conditions of the Maximum Principle for the performance index

$$J(\underline{x}_o, t_o, T; v) = \frac{1}{2}\,\underline{x}'(T)\;\underline{Q}\;\underline{x}(T)$$

if and only if

$$\underline{D}\;\underline{F}_{\alpha}(\underline{x}_o, t_o, \underline{\alpha}) = 2\;F(\underline{x}_o, t_o, \underline{\alpha})\;\underline{\alpha}$$

where $\underline{\alpha} \in \Omega^r$. For the proof see Theorems 2.1 and 2.2.

4. $$\max_{v \in S_v} J(\underline{x}_o, t_o, T; v) = \max_{\underline{\alpha} \in \Omega^r} F(\underline{x}_o, t_o, \underline{\alpha})$$

For the proof see Theorem 2.3.

5. $F(\underline{x}_o, t_o, \underline{\alpha})$ is a convex function of $\underline{\alpha} \in E^n$. The proof follows from an application of the definition of convexity (see Section 1.6) to the expression in (2.30).

6. $F(\underline{x}_o, t_o, \underline{\alpha}) = F(\underline{x}_o, t_o, -\underline{\alpha})$, $\underline{\alpha} \in E^n$. This may be seen from an inspection of (2.26).

7. $J(\underline{x}_o, t_o, T; v(t,\underline{\alpha})) \geq F(x_o, t_o, \underline{\alpha})$, $\underline{\alpha} \in \Omega^r$. This follows from (2.55).

8. $F(\underline{x}_o, t_o, \underline{\alpha}) = \frac{1}{2}\,F'_{\alpha}(\underline{x}_o, t_o, \underline{\alpha})'\;\underline{\alpha}$, $\underline{\alpha} \in \Omega^r$. This follows from (2.40).

9. If $\underline{D}\;F_{\alpha}(\underline{x}_o, t_o, \underline{\alpha}) = 2\;F(\underline{x}_o, t_o, \underline{\alpha})\;\underline{\alpha}$, $\underline{\alpha} \in \Omega^r$, then the following hold:

   a) $$\underline{\alpha} = \frac{\pm\,\underline{D}(\underline{M}^{-1})'\;\underline{x}(T,\underline{\alpha})}{||\underline{D}\;\underline{M}'^{-1}\;\underline{x}(T,\underline{\alpha})||}$$

This is easily proved using (2.43) and Property 6 above.

b) $J(\underline{x}_o, t_o, T; v(t,\underline{\alpha})) = F(\underline{x}_o, t_o, \underline{\alpha})$

This follows from Property 9a, and equation (2.54).

c) $\underline{F}_{\underline{\alpha}}(\underline{x}_o, t_o, \underline{\alpha})' \, \underline{D} \, \underline{F}_{\underline{\alpha}}(\underline{x}_o, t_o, \underline{\alpha}) = [2 \, F(\underline{x}_o, t_o, \underline{\alpha})]^2$

This follows directly from the hypothesis above.

For computational purposes one might make the substitution

$$\alpha_r = (1 - \sum_{i=1}^{r-1} \alpha_i^2)^{1/2} \tag{2.62}$$

and consider F as a function of the independent variables α_1 to α_{r-1}. Assuming the substitution in (2.62) has been made, it is convenient to define the gradient of F, $\nabla^r \underline{F}$ as an r-1 vector with i^{th} component

$$\nabla^r F_i = F_{\alpha_i} + F_{\alpha_r} \frac{d\alpha_r}{d\alpha_i} \qquad i = 1 \text{ to } r-1$$

But from (2.62)

$$\frac{d\alpha_r}{d\alpha_i} = - \frac{\alpha_i}{\alpha_r}$$

Thus

$$\nabla^r F_i = F_{\alpha_i} - F_{\alpha_r} \frac{\alpha_i}{\alpha_r} \qquad i = 1 \text{ to } r-1 \tag{2.63}$$

We may now state the following additional properties:

10. If $\underline{D} \, \underline{F}_{\underline{\alpha}}(\underline{x}_o, t_o, \underline{\alpha}) = 2 \, F(\underline{x}_o, t_o, \underline{\alpha}) \, \underline{\alpha}$, $\underline{\alpha} \in \Omega^r$, then $\nabla^r \underline{F} = 0$.

    The proof of this follows directly from (2.63).

11. The vector $\nabla^r \underline{F}$ is discontinuous on the surface described by

$$\left.\begin{aligned} \xi_r \alpha_r &= - \sum_{i=1}^{r-1} \alpha_i \xi_i \\ \alpha_r &= (1 - \sum_{i=1}^{r-1} \alpha_i^2)^{1/2} \end{aligned}\right\} \tag{2.64}$$

where

$$\underline{\xi} = \underline{P}'(t_o)\,\underline{x}_o + \int_{t_o}^{T} \frac{[\gamma_2(\tau) + \gamma_1(\tau)]}{2}\,\underline{P}'(\tau)\,\underline{c}(\tau)\,d\tau \tag{2.65}$$

To prove this, it is noted from (2.30) that $\underline{F}_\alpha$ has a discontinuity along the surface $\underline{\xi}'\underline{\alpha} = 0$. Property 11 follows directly. It may be readily determined that the expressions in (2.64) describe a half hyper-ellipsoidal surface in r-1 space. The equation of the entire ellipsoid is

$$\sum_{i=1}^{r-1} (\xi_r^2\,\alpha_i^2 + \sum_{j=1}^{r-1} \xi_i\xi_j\,\alpha_i\alpha_j) = \xi_r^2 \tag{2.66}$$

This surface divides the r-1 dimensional space on which F is defined into two parts.

These properties are best illustrated by means of examples.

Example 2.6.1

Consider the system

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = v$$

where

$$|v| \leq 1$$

and

$$\left.\begin{aligned} x_1(0) &= 3.5 \\ x_2(0) &= -1 \end{aligned}\right\}$$

The problem is to maximize the performance index

$$J(\underline{x}(0),\ 0,\ 3;\ v) = \frac{1}{2}\,[x_1^2(3) + x_2^2(3)]$$

with respect to the disturbance v. It is easily determined that

$$\underline{P}(t) = \begin{pmatrix} 1 & 0 \\ 3-t & 1 \end{pmatrix}$$

$$\theta(t,\underline{\alpha}) \equiv 0$$

$$\phi(t,\underline{\alpha}) = \int_t^3 |(3-\tau)\,\alpha_1 + \alpha_2|\, d\tau$$

Thus

$$F(\underline{x}(t),\, t,\, \underline{\alpha}) = \frac{1}{2}\,[\,|\alpha_1\, x_1(t) + [(3-t)\,\alpha_1 + \alpha_2]\, x_2(t)| + \phi(t,\underline{\alpha})]^2$$

A plot of $F(x(t_o),\, t_o,\, \underline{\alpha})$ vs. α_1 for

$$\alpha_2 = (1 - \alpha_1^2)^{1/2}$$

is shown in Figure 2.1. The other curve in the figure is $J(x_o,\, t_o,\, T;\, v(t,\underline{\alpha}))$. The values of α_1 resulting in $\nabla^r \underline{F} = 0$ are $-1/\sqrt{5}$, $1/\sqrt{2}$ and $5/\sqrt{29}$. The corresponding final states and values of performance index are given in Table 2.1.

Table 2.1 Results for Example 2.6.1

α_1	$x_1(3)$	$x_2(3)$	J
$-1/\sqrt{5}$	1	-2	2.5
$1/\sqrt{2}$	-4	-4	16.0
$5/\sqrt{29}$	5	2	14.5

It can be easily verified that <u>each</u> of the final states above results from a trajectory satisfying the Maximum Principle. Since the above data exhausts all possibilities, it is clear that the worst case value of the performance index (i.e., the global maximum) is $J = 16$.

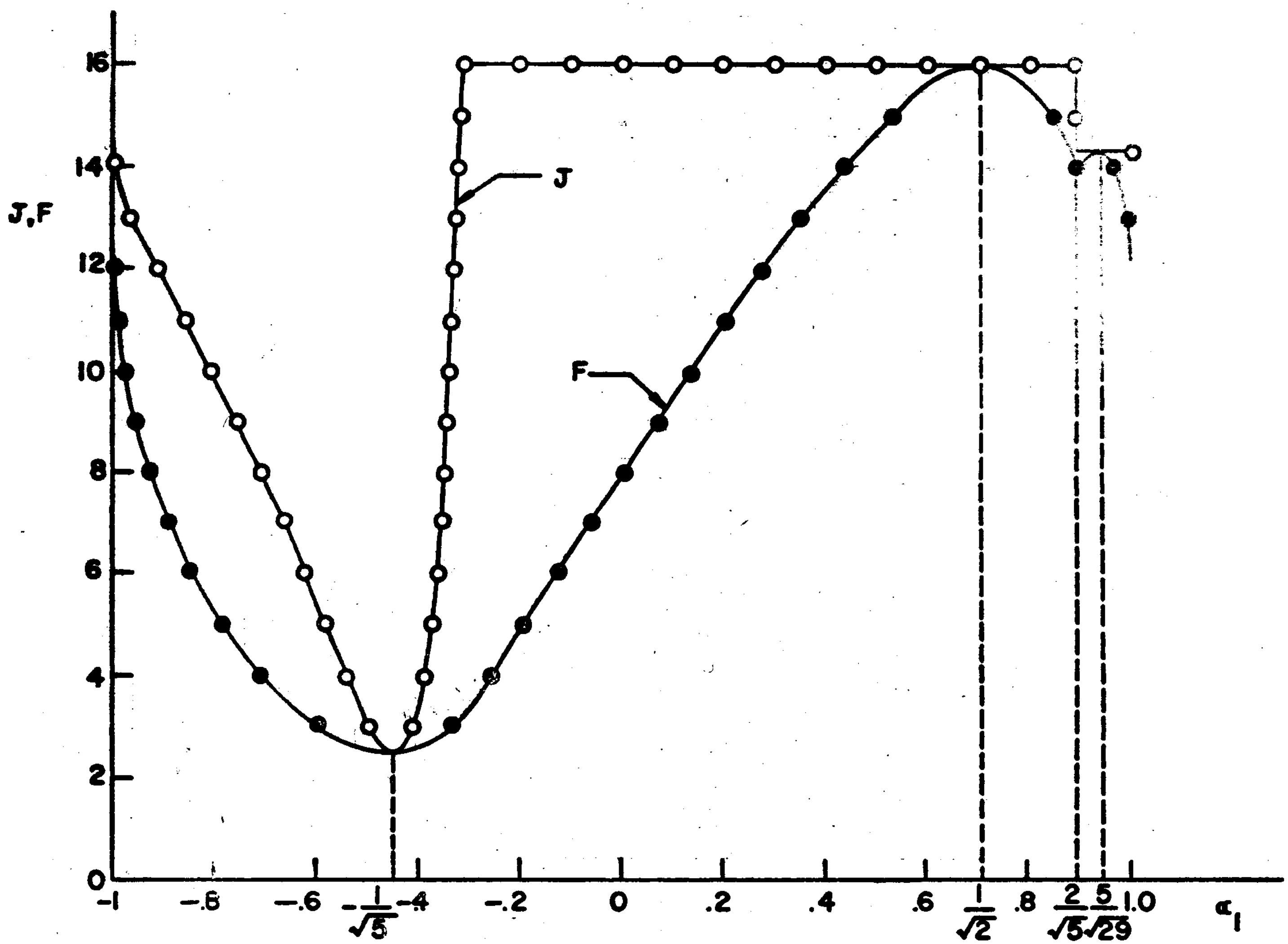

Figure 2.1. $J(\underline{x}_o, 0, 3; v(t,\underline{\alpha}))$ and $F(\underline{x}_o, 0, \underline{\alpha})$ vs α_1 for Example 2.6.1.

Note that Property 11 implies that $\nabla^r F$ will have a discontinuity at $\alpha_1 = 2/\sqrt{5}$. Inspection of the curve for F in Figure 2.1 indicates that this is indeed the case.

Example 2.6.2

Consider the third-order system described by

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = x_3$$

$$\dot{x}_3 = -8x_1 - 6x_2 - 3x_3 + 8v$$

where

$$|v| \leq 1$$

For a given initial state $\underline{x}_o$, the problem is to maximize

$$J(\underline{x}_o, 0, T; v) = \frac{1}{2}[x_1^2(T) + x_2^2(T) + x_3^2(T)]$$

with respect to v.

This problem has been solved for two different initial states and for two different values of T. The results are summarized in Table 2.2 below.

Table 2.2 Results for Example 2.6.2

Hill	Initial State $(\underline{x}_o')$	T	α_1^*	α_2^*	$J(\underline{x}_o, 0, T; v^*)$
1	(0, 0, 0)	1	-.14	-.06	4.33
2	(10, -5, 1)	1	.18	-.81	64.6
3	(0, 0, 0)	3	-.20	-.05	20.2
4	(10, -5, 1)	3	-.16	-.31	53.3

Level curves (curves of constant $F(\underline{x}_o, 0, \underline{\alpha})$) for "hills" 1 to 4 on the (α_1, α_2) plane are shown in Figures 2.2 to 2.5. The dependent

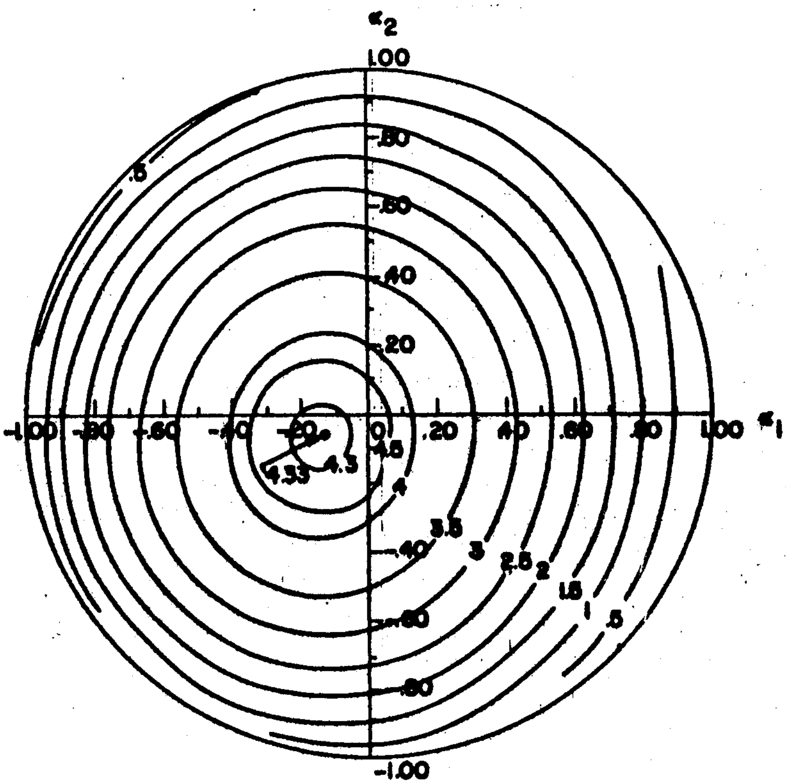

Figure 2.2. Level curves for Hill 1.

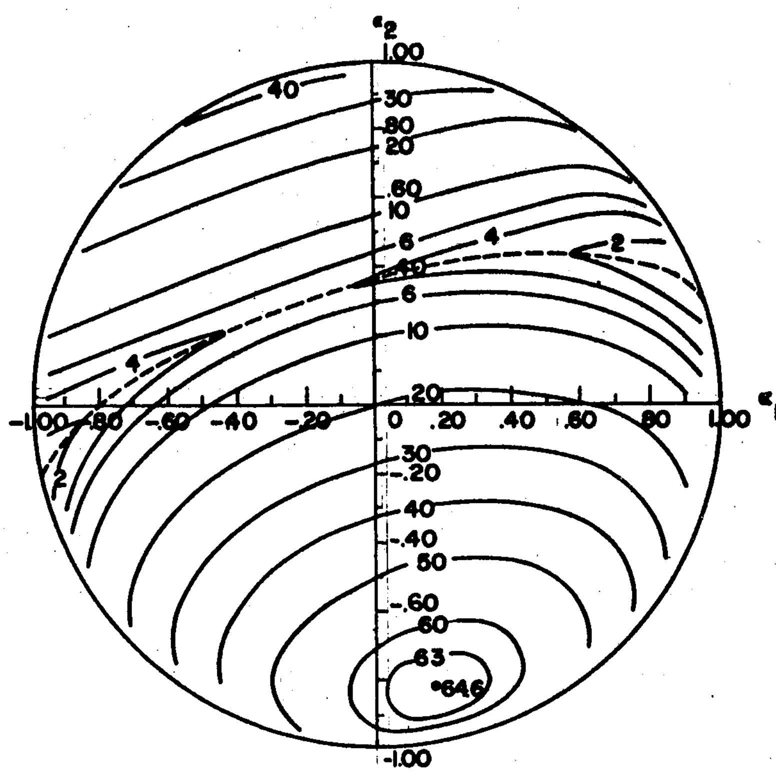

Figure 2.3. Level curves for Hill 2.

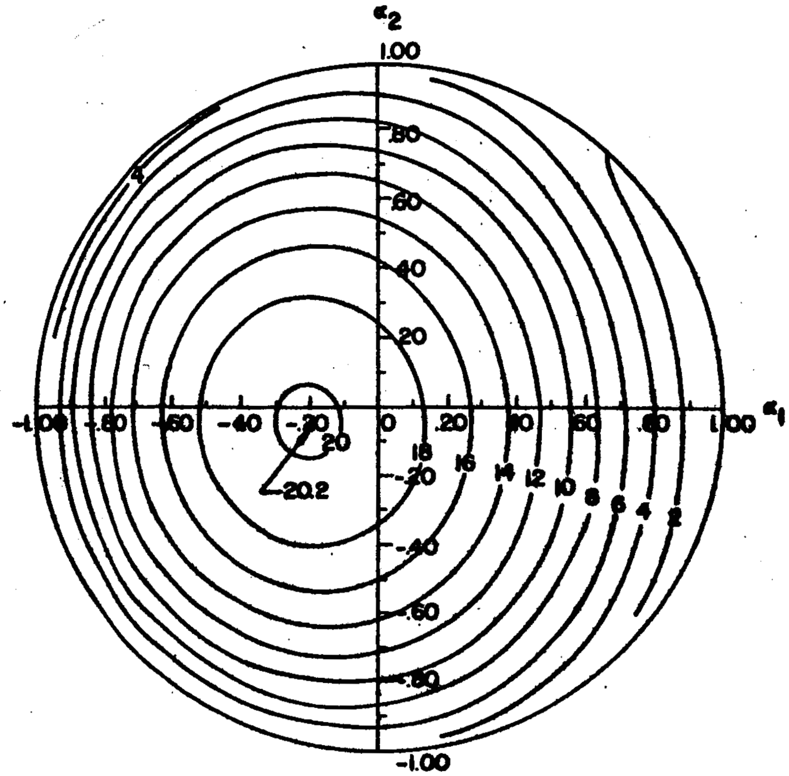

Figure 2.4. Level curves for Hill 3.

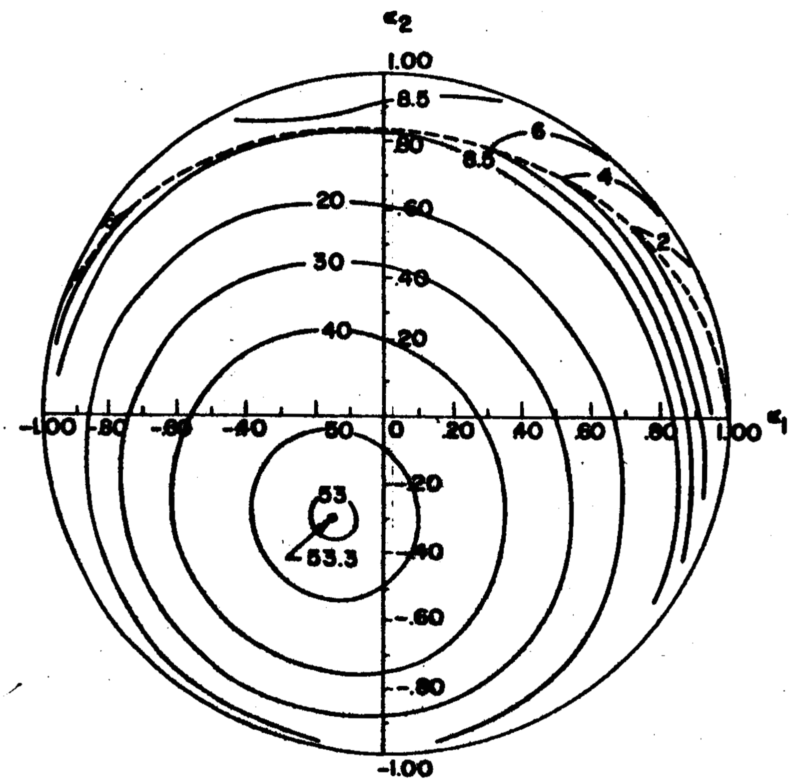

Figure 2.5. Level curves for Hill 4.

variable

$$\alpha_3 = (1 - \alpha_1^2 - \alpha_2^2)^{1/2}$$

It is observed that hills 2 and 4 are multimodal. The elliptical surfaces of discontinuity of $\nabla^r F(\underline{x}_o, 0, \underline{\alpha})$ are clearly visible.

Note that the points at which the peaks of hills 3 and 4 occur are closer together than the corresponding points for hills 1 and 2. In addition it is observed that the surface of discontinuity for hill 4 is closer to the bounding unit circle than is the surface of discontinuity for hill 2. In fact as $T \to \infty$, one would expect α^* to become independent of the initial state, and the surface of discontinuity to approach the unit circle. The reader may easily convince himself (see equation 2.65) that these statements are true for any stable system with a symmetrically bounded forcing function.

Figures 2.2 and 2.4 seem to indicate that hills 1 and 3 are unimodal.† This is primarily due to the incomplete data presentations in the proximity of the bounding unit circles. A second plot of level curves for hill 1 (Figure 2.6) in the (α_1, α_3) plane with

$$\alpha_2 = \sqrt{1 - \alpha_1^2 - \alpha_3^2}$$

clearly indicates that hill 1 is multimodal.

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†A unimodal hill is a hill with a single peak.

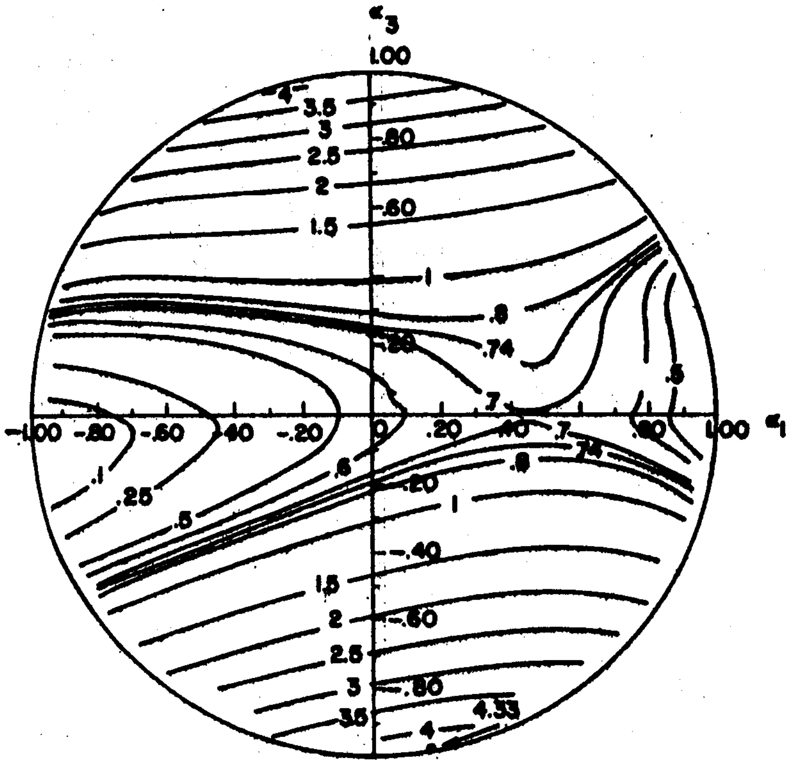

Figure 2.6. Level curve for Hill 1 using α_1 and α_3 as independent variables.

2.7 Geometric Interpretation

It is possible to give a geometric interpretation to the function $F(\underline{x}_o, t_o, \underline{\alpha})$ used in Example 2.6.1. To do this first consider

$$E(T,\underline{\alpha}) = |\alpha_1 x_1(3) + \alpha_2 x_2(3)|$$

which is the error function for the example. $E(T,\underline{\alpha})$ may be thought of as the magnitude of the projection of the final state $\underline{x}(3)$ in the $\underline{\alpha}$ direction (remember that $\underline{\alpha}$ has unit length). It is clear that the worst case value of $E(T,\underline{\alpha})$ is determined by the final state $\underline{x}(3)$ which has the largest projection in the $\underline{\alpha}$ direction.

To proceed further, we shall make use of the notion of a reachable zone [4, 7]. A T-second reachable zone for a forced linear system consists of all the final states which may be reached from the given initial state in T seconds due to the system forcing function. The three second reachable zone for Example 2.6.1 with zero initial conditions has been previously determined [4] and is shown in Figure 2.7. It may be readily established that the homogeneous part of the final state (i.e., that part of the final state due just to the initial conditions) for the example is

$$\underline{x}_h = \begin{pmatrix} .5 \\ -1 \end{pmatrix}$$

This means that the reachable zone for the initial conditions in the problem may be determined by shifting the center of the zone shown in Figure 2.7 to the point $\underline{x}_h$. The resulting reachable zone is shown in

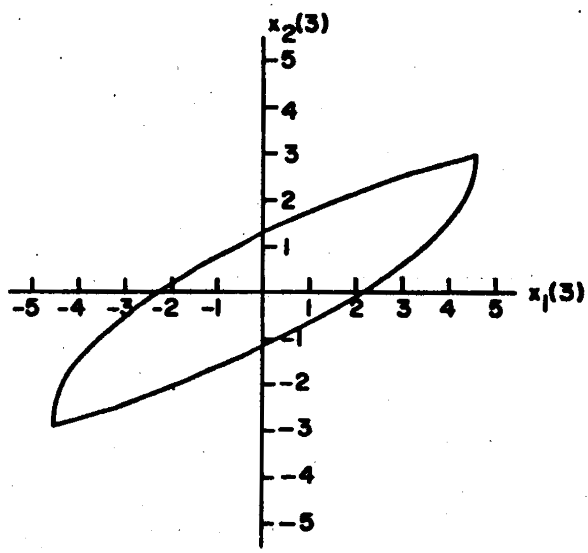

Figure 2.7. Three-second reachable zone for the system in Example 2.6.1 with zero initial conditions.

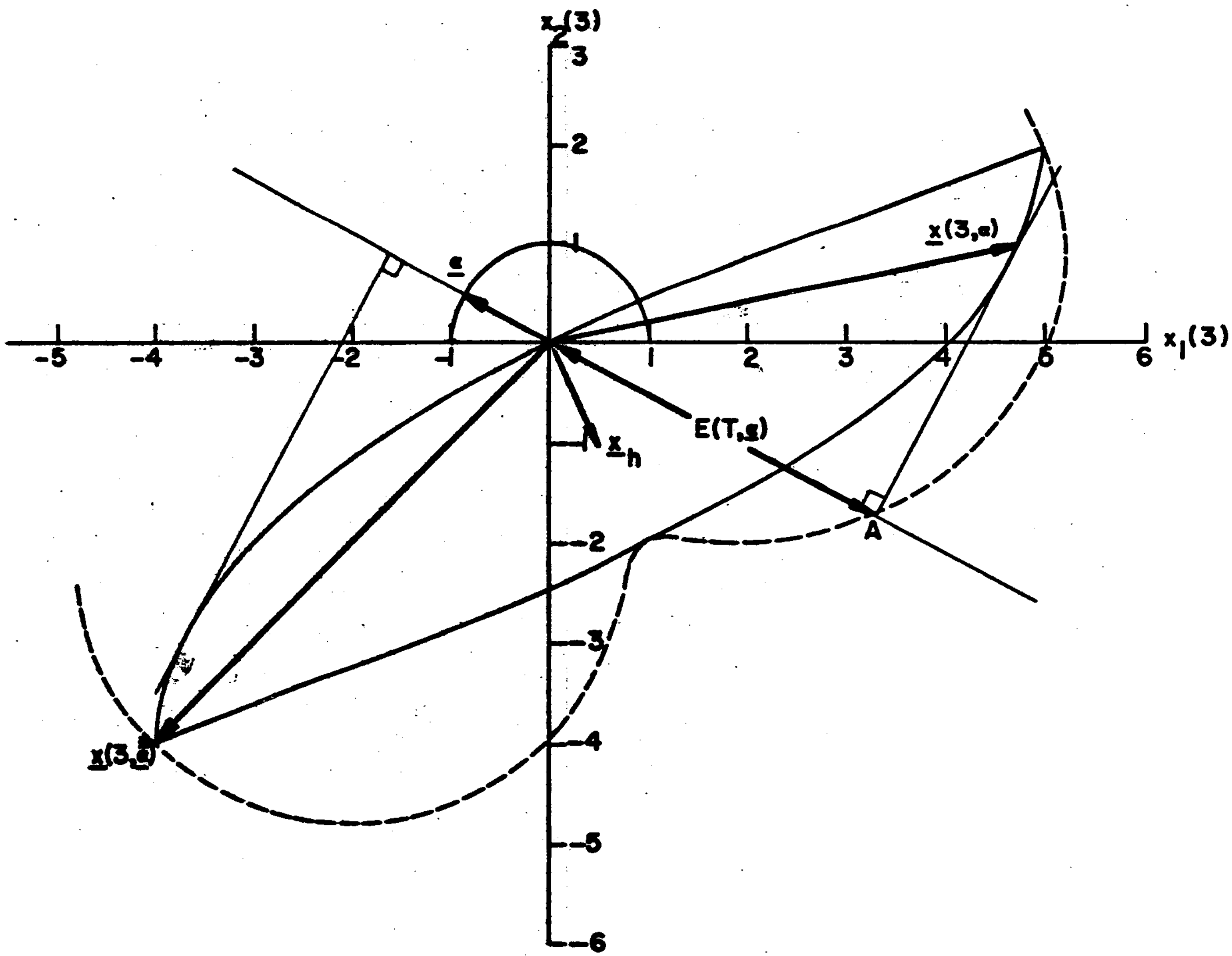

Figure 2.8. Construction of $F(\underline{x}_o, t_\sigma, \underline{\alpha})$ for Example 2.6.1.

Figure 2.8. (The fact that the boundary of this zone intersects the origin is accidental and of no relevance.)

How to determine the worst case value of the error function is now apparent. First the line in the $\underline{\alpha}$ direction is extended. Then normals to this line which are tangent to the boundaries of the reachable zone are determined. The worst case value of $E(t,\underline{\alpha})$ is clearly equal to the larger of the two final state projections in the $\underline{\alpha}$ direction. The corresponding worst case final state is indicated by $\underline{x}(3,\alpha)$. Thus for the given value of $\underline{\alpha}$, the distance from the origin to point A in Figure 2.8 represents the worst case value of the error function. Since $F(\underline{x}_o, t_o, \underline{\alpha})$ is equal to one-half of the square of the value of the worst case error function, the interpretation of $F(\underline{x}_o, t_o, \underline{\alpha})$ is complete.

The operations just described can be performed for many values of $\underline{\alpha}$ resulting in the dashed line shown in the figure. The dashed line touches the boundary of the reachable zone for the three values of α_1 given in Table 2.1.

Though the idea of a reachable zone is helpful in understanding the $F(\underline{x}_o, t_o, \underline{\alpha})$ function, reachable zone theory is of limited value as a means of solution of the general worst case error problem. This is because for systems of order higher than two the computation of a reachable zone boundary is quite difficult.

2.8 Computational Considerations

As noted in a previous section, the maximization of $F(\underline{x}_o, t_o, \underline{\alpha})$ is a multimodal hill-climbing problem. Each local maximum of the hill corresponds to a solution of the Maximum Principle equations. At the present time, the author knows of no method (short of an exhaustive search in $\underline{\alpha}$ space) which is guaranteed to find the global maximum of a multimodal hill.

One method which might be tried is outlined as follows. First define a set of evenly spaced points in Ω^r. From each of these points use an efficient local hill-climbing technique to determine the local maximum. Compare the local solutions to determine the best possible candidate for the global solution. Because $F(\underline{x}_o, t_o, \underline{\alpha})$ is usually a relatively smooth function of $\underline{\alpha}$ (not too many peaks) and because Ω^r is compact, such an approach is feasible if r is not too large. Of course, an efficient local hill-climbing method is required.

We shall suggest two computational techniques for determining a local maximum of the hill. Both are iterative methods.

The functions $\underline{F}_{\alpha}(\underline{x}_o, t_o, \underline{\alpha})$ and $F(\underline{x}_o, t_o, \underline{\alpha})$ are utilized in both computational schemes. It is possible to compute these quantities on successive iterations without solving any differential equations. Consider the expression

$$\underline{F}_{\alpha}(\underline{x}_o, t_o, \underline{\alpha}) = \Big\{\underline{\xi}\ \mathrm{sgn}(0, \underline{\alpha}'\underline{\xi}) + \int_{t_o}^{T} \left[\frac{\gamma_2(t) - \gamma_1(t)}{2} \underline{P}'(t)\ \underline{c}(t)\right]\left[\mathrm{sgn}(0, \underline{c}'(t)\ \underline{P}(t)\ \underline{\alpha}\right] dt\Big\}\ E(T,\underline{\alpha}) \tag{2.67}$$

which follows from (2.39) and (2.65). If we define the n-vector

$$\underline{w}(t) = \int_{t_o}^{t} \left[\frac{\gamma_2(\tau) - \gamma_1(\tau)}{2} \right] \underline{P}'(\tau)\ \underline{c}(\tau)\ d\tau \tag{2.68}$$

it may be readily determined from (2.67) that

$$\frac{\underline{F}_\alpha(\underline{x}_o,\ t_o,\ \underline{\alpha})}{E(T,\underline{\alpha})} = \Big[\underline{\xi}\ \mathrm{sgn}(0,\ \underline{\alpha}'\underline{\xi}) + \underline{w}(t_1)\ \mathrm{sgn}(0,\ \underline{c}(\bar{t}_1)\ \underline{P}(\bar{t}_1)\ \underline{\alpha})$$

$$+ \sum_{i=1}^{k} (\underline{w}(t_{i+1}) - \underline{w}(t_i))\ \mathrm{sgn}(0,\ \underline{c}'(\bar{t}_{i+1})\ \underline{P}(\bar{t}_{i+1})\ \underline{\alpha}\Big] \tag{2.69}$$

where the t_i's, i=1 to k, are the zeros of $(\underline{c}'(t)\ \underline{P}(t)\ \underline{\alpha})$ (or the end-points of intervals on which $\underline{c}'\underline{P}\underline{\alpha} \equiv 0$) on (t_o,T) and $t_{k+1} = T$. The $\bar{t}_i$'s, i = 1 to k+1, are arbitrary points satisfying $\bar{t}_i \in (t_{i-1},\ t_i)$. Note that since $\underline{w}(t)$ and $\underline{\xi}$ are independent of $\underline{\alpha}$, they need only be computed once.

To obtain $F(\underline{x}_o,\ t_o,\ \underline{\alpha})$ and $\underline{F}_\alpha(\underline{x}_o,\ t_o,\ \underline{\alpha})$ from (2.69) is a simple matter. We know from Property 8 in Section 2.6 that

$$F(\underline{x}_o,\ t_o,\ \underline{\alpha}) = \frac{\underline{F}'_\alpha(\underline{x}_o,\ t_o,\ \underline{\alpha})\ \underline{\alpha}}{2} \tag{2.70}$$

But since

$$F(\underline{x}_o,\ t_o,\ \underline{\alpha}) = \frac{E^2(T,\underline{\alpha})}{2} \tag{2.71}$$

it is clear that

$$E(T,\underline{\alpha}) = \frac{\underline{F}'_\alpha(\underline{x}_o,\ t_o,\ \underline{\alpha})\ \underline{\alpha}}{E(T,\underline{\alpha})} \tag{2.72}$$

Thus using (2.69) and (2.72), $E(T,\underline{\alpha})$ can be computed. Using (2.69) again and (2.71), one can compute $\underline{F}_\alpha(\underline{x}_o,\ t_o,\ \underline{\alpha})$ and $F(\underline{x}_o,\ t_o,\ \underline{\alpha})$. Note that except for the initial computations needed to compute $\underline{P}'(t)\underline{c}$, $\underline{w}(t)$ and $\underline{\xi}$, no integration is required.

It is also pointed out that only the first r elements of $\underline{F}_\alpha(\underline{x}_o,\ t_o,\ \underline{\alpha})$ are required for computation. Hence, rather than computing and storing (if a digital computer is used) all the elements in $\underline{\xi}$, $\underline{w}(t)$, and $P'(t)\ \underline{c}(t)$, one need only compute and store the first r elements. Thus memory requirements are roughly 2r time functions.

The first computational method to be considered is simply a gradient technique. Basically the policy is to change $\underline{\alpha}$ in the direction of steepest ascent on the hill. Given the point $\underline{\alpha}^j$, to determine $\underline{\alpha}^{j+1}$ so that

$$F(\underline{x}_o, t_o, \underline{\alpha}^{j+1}) \geq F(\underline{x}_o, t_o, \underline{\alpha}^j)$$

one could use the scheme

$$\alpha_i^{j+1} = \alpha_i^j + S \ \nabla^r F(\underline{x}_o, t_o, \underline{\alpha}^j) \qquad i = 1 \text{ to } r-1 \tag{2.73}$$

$$\alpha_r^{j+1} = \left(1 - \sum_{i=1}^{r-1} \alpha_i^{j+1} \right)^{1/2}$$

where S is a positive number (the step size). The gradient components $\nabla^r F_i$ may be computed using the expression in (2.63). The standard method for determining S is to assume that ΔF, the change in F,

$$\Delta F(\underline{\alpha}^{j+1}) = F(\underline{x}_o, t_o, \underline{\alpha}^{j+1}) - F(\underline{x}_o, t_o, \underline{\alpha}^j) \tag{2.74}$$

satisfies

$$\Delta F(\underline{\alpha}^{j+1}) \approx \sum_{i=1}^{r-1} (\alpha_i^{j+1} - \alpha_i^j) \nabla^r F_i(\underline{x}_o, t_o, \underline{\alpha}^j) \tag{2.75}$$

By then specifying the desired fractional change

$$\rho^{j+1} = \frac{\Delta F(\underline{\alpha}^{j+1})}{F(\underline{x}_o, t_o, \underline{\alpha}^j)} \tag{2.76}$$

one may solve for S.

$$S = \frac{\rho^{j+1} F(\underline{x}_o, t_o, \underline{\alpha}^j)}{\sum_{i=1}^{r-1} (\nabla^r F_i(\underline{x}_o, t_o, \underline{\alpha}^j))^2} \tag{2.77}$$

Of course for (2.75) to hold, one must pick ρ^{j+1} small. Just how small can only be determined by trial and error.

We have tried this approach on several problems with moderate success. The main problem has been (as it usually is with gradient techniques) the determination of a suitable step size.

Rather than pursue the gradient method any further, we consider a second method which completely avoids the step size problem. It has been noted in Section (2.6) that $F(\underline{x}_o, t_o, \underline{\alpha})$ is a convex function of $\underline{\alpha}$. Thus using the property of convex functions described in (1.6)

$$F(\underline{x}_o, t_o, \underline{\alpha}^{j+1}) \geq F(\underline{x}_o, t_o, \underline{\alpha}^j) + \underline{F}'_\alpha(\underline{x}_o, t_o, \underline{\alpha}^j)\,(\underline{\alpha}^{j+1} - \underline{\alpha}^j) \tag{2.78}$$

Thus

$$\Delta F(\underline{\alpha}^{j+1}) \geq \underline{F}'_\alpha(\underline{x}_o, t_o, \underline{\alpha}^j)\,(\underline{\alpha}^{j+1} - \underline{\alpha}^j) \tag{2.79}$$

The expression on the right will be a maximum for $\underline{\alpha}^{j+1} \epsilon\, \Omega^r$ if

$$\underline{\alpha}^{j+1} = \frac{\underline{F}'_\alpha(\underline{x}_o, t_o, \underline{\alpha}^j)}{||\underline{F}_\alpha(\underline{x}_o, t_o, \underline{\alpha}^j)||} \tag{2.80}$$

Furthermore, $\Delta F(\underline{\alpha}^{j+1})$ will be strictly positive if $\underline{\alpha}^j \neq \underline{\alpha}^{j+1}$, $\underline{\alpha}^j \epsilon\, \Omega^r$. If $\underline{\alpha}^{j+1} = \underline{\alpha}^j$, it follows from Property 3, Section (2.6) that a local maximum of F has been found.† Thus using the policy expressed in (2.80), convergence to a local maximum is assured.

This computational technique has been applied to several problems including Examples 2.6.1 and 2.6.2. Using different initial guesses for $\underline{\alpha}^1$, convergence to local maxima was invariably rapid (less than 5

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†Of course, one might make an initial guess for $\underline{\alpha}$ corresponding to a local minimum of F. For such a case, using (2.80) $\underline{\alpha}^2 = \underline{\alpha}^1$. If, however, the initial guess is not right at a minimum point, $\underline{\alpha}^{j+1} = \underline{\alpha}^j$ can only occur at a local maximum.

iterations). In view of the fact that the amount of computation per iteration is very small, this technique is considered to be highly efficient.

2.9 Summary

In this chapter a general error analysis problem for linear systems with quadratic error criteria and bounded disturbances has been formulated. By identifying this general problem with a simpler one, we have been able to show that the maximum of the functional $J(\underline{x}_o, t_o, T; v)$ with respect to the forcing function $v(t)$ is equal to the maximum of the function $F(\underline{x}_o, t_o, \underline{\alpha})$ with respect to $\underline{\alpha} \in \Omega^r$. A geometric interpretation of the function $F(\underline{x}_o, t_o, \underline{\alpha})$ has also been presented.

Since the dimensionality of Ω^r depends only on the rank of $\underline{Q}$ and not on the order of the system, high order systems present no special computational difficulties. This reduction of dimensionality was suggested in [12].

In general, if the system is not completely controllable (for a definition of controllability, see [24]) with respect to the disturbance, singular surfaces can exist [4]. Since our results take into account singular cases, a controllability assumption (which is unreasonable for disturbance inputs) has not been required. It has been noted that while on a singular surface, the disturbance has no effect on the performance index.

The multimodal nature of $F(\underline{x}_o, t_o, \underline{\alpha})$ has been established. It was pointed out that each local maximum of $F(\underline{x}_o, t_o, \underline{\alpha})$ corresponds to a local solution of the Maximum Principle equations. However, since $F(\underline{x}_o, t_o, \underline{\alpha})$ is relatively smooth and since Ω^r is compact, for small r one should be able to determine the maximum with a reasonable amount of effort.

To aid in computing the maximum, an efficient computational algorithm has been presented. It should be mentioned that other computational techniques have been proposed for solving the problem formulated in this chapter [28], [4]. The author's experience with these other techniques has indicated that they require considerably more computation and are not as efficient as the scheme proposed in this chapter.

CHAPTER 3

A STUDY OF MIN-MAX PROBLEMS FOR LINEAR SYSTEMS

3.1 Introduction

In the design of a control system based on a worst case disturbance approach, one attempts to determine a feedback controller which will result in an acceptably small worst case value of a system error function. Thus for each controller considered in the design, a worst case disturbance error analysis is performed to determine if the control is acceptable. Often, however, one is interested in having the system behave not only in an acceptable manner, but also in the best possible manner. In view of this, it seems reasonable to pose the following problem: Determine the feedback controller from a suitably defined class which will result in the smallest worst case disturbance value of the system error function. Mathematically, this amounts to determining the controller which minimizes the system error function while the disturbance acts to maximize it. This problem shall be referred to as the "min-max" problem.

On the following pages, a more detailed formulation of the problem is presented. The relationship between the min-max problem and the so-called "differential game" is indicated. In the remainder of the chapter, the results of an investigation of the min-max problem are described. By considering general linear systems and specific performance indices, it has been possible to make a complete analysis of several problems.

3.2 Formulation of the Min-Max Problem

In this section, a min-max problem is formulated for a general nonlinear dynamic system. The formulation begins by specifying the system vector differential equation

$$\underline{\dot{x}}(t) = \underline{f}(\underline{x}, t, \underline{u}, \underline{v}) \tag{3.1}$$

where $\underline{x}$ is an n-vector of state variables for the system, $\underline{u}(t)$ is an m-vector control, and $\underline{v}(t)$ is an r-vector disturbance acting on the system. The system is to be judged in terms of the performance index

$$J(\underline{x}_o, t_o, T; \underline{u}, \underline{v}) = \int_{t_o}^{T} g(\underline{x}, t, \underline{u}, \underline{v})\, dt + G(\underline{x}(T),T) \tag{3.2}$$

where g and G are scalar functions. The initial state $\underline{x}_o$ is arbitrary but fixed. The initial and terminal times t_o and T are also fixed.

Let $S_{\underline{u}}$ and $S_{\underline{v}}$ be suitably defined constraint sets for $\underline{u}$ and $\underline{v}$. We shall consider the controller $\underline{U}(\underline{x},t)$ to be a member of the class $C_{\underline{u}}$ if along a trajectory $\underline{x}(t)$ which is a solution of

$$\underline{\dot{x}}(t) = \underline{f}(\underline{x}, t, U(\underline{x},t), \underline{v}) \tag{3.3}$$

$\underline{v} \in S_v$, the relationship

$$\underline{u}(t) \equiv \underline{U}(\underline{x}(t),t) \tag{3.4}$$

results in $\underline{u}(t) \in S_{\underline{u}}$. It is assumed that if $\underline{U}(\underline{x},t) \in C_{\underline{u}}$, then

$$\max_{\underline{v} \in S_{\underline{v}}} J(\underline{x}_o, t_o, T; \underline{U}(\underline{x},t), \underline{v}) \tag{3.5}$$

exists. Maximization here, is meant in the same sense as in Chapter 2. That is, the disturbance $\underline{v}(t)$ maximizes the functional in (3.5) subject to the differential equation in (3.3). Thus the quantity in (3.5) represents the worst case disturbance value of the performance index in (3.2) for the controller $\underline{U}(\underline{x},t)$.

With these preliminaries completed, the min-max problem may now be stated as follows: Determine a controller $\underline{u}^* = \underline{U}^*(\underline{x},t) \in C_{\underline{u}}$ such that

$$\max_{\underline{v} \in S_{\underline{v}}} J(\underline{x}_o, t_o, T; \underline{U}^*(\underline{x},t), \underline{v}) \leq \max_{\underline{v} \in S_{\underline{v}}} J(\underline{x}_o, t_o, T; \underline{U}(\underline{x},t), \underline{v}) \qquad (3.6)$$

for all $\underline{U}(\underline{x},t) \in C_{\underline{u}}$. Thus the problem is to determine a controller $\underline{U}^*(\underline{x},t)$ which will result in the smallest worst case value of the performance index in (3.2).

It is noted that the problem implied by (3.5) is a one sided maximization problem. It is convenient for analysis purposes to think of the maximizing disturbance as a function of state. Actually this function will also depend on $\underline{U}(\underline{x},t)$. Hence in defining a worst case disturbance feedback function, a particular $\underline{U}(\underline{x},t)$ must be specified. Define $\underline{v}^* = \underline{V}^*(\underline{x},t)$ as the maximizing disturbance for the functional

$$J(\underline{x}_o, t_o, T; \underline{U}^*(\underline{x},t), \underline{v}) \qquad (3.7)$$

With these definitions in mind, the following relationships should be self-evident.

$$\min_{\underline{u} \in S_{\underline{u}}} \max_{\underline{v} \in S_{\underline{v}}} J(\underline{x}_o, t_o, T; \underline{u}, \underline{v}) = \max_{\underline{v} \in S_{\underline{v}}} J(\underline{x}_o, t_o, T; \underline{u}^*, \underline{v}) \qquad (3.8)$$

$$\min_{\underline{u} \in S_{\underline{u}}} \max_{\underline{v} \in S_{\underline{v}}} J(\underline{x}_o, t_o, T; \underline{u}, \underline{v}) = J(\underline{x}_o, t_o, T; \underline{u}^*, \underline{v}^*) \qquad (3.9)$$

3.3 The Relationship Between the Min-Max Problem and the Differential Game

In spite of its intuitive appeal, the min-max problem formulated in the last section has received almost no attention in the literature.

However, a closely related problem, the so-called differential game, has been the subject of much investigation in recent years [13-20].

In the differential game one attempts to determine "strategies" $\hat{\underline{U}}(\underline{x},t)$ and $\hat{\underline{V}}(\underline{x},t)$ from suitably defined classes of functions $\{\underline{U}(\underline{x},t)\}$ and $\{\underline{V}(\underline{x},t)\}$ so that

$$J(\underline{x}_o, t_o, T; \hat{\underline{U}}(\underline{x},t), \hat{\underline{V}}(\underline{x},t)) \leq J(\underline{x}_o, t_o, T; \underline{U}(x,t), \hat{\underline{V}}(\underline{x},t)) \tag{3.10}$$

and

$$J(\underline{x}_o, t_o, T; \hat{\underline{U}}(\underline{x},t), \hat{\underline{V}}(\underline{x},t)) \geq J(\underline{x}_o, t_o, T; \hat{\underline{U}}(x,t), \underline{V}(\underline{x},t)) \tag{3.11}$$

A pair of strategies satisfying (3.10) and (3.11) is said to represent a "saddle point."

It is easily shown that the differential game is a min-max problem. Using (3.10)

$$J(\underline{x}_o, t_o, T; \hat{\underline{U}}(\underline{x},t), \hat{\underline{V}}(\underline{x},t)) \leq \max_{\underline{v} \in S_{\underline{v}}} J(\underline{x}_o, t_o, T; \underline{U}(\underline{x},t), \underline{v}) \tag{3.12}$$

It is clear from (3.11) that

$$J(\underline{x}_o, t_o, T; \hat{\underline{U}}(\underline{x},t), \hat{\underline{V}}(\underline{x},t)) = \max_{\underline{v} \in S_{\underline{v}}} J(\underline{x}_o, t_o, T; \hat{\underline{U}}(\underline{x},t), \underline{v}) \tag{3.13}$$

Combining (3.12) and (3.13) one finds that

$$\max_{\underline{v} \in \underline{S}_{\underline{v}}} J(\underline{x}_o, t_o, T; \hat{U}(\underline{x},t), \underline{v}) \leq \max_{\underline{v} \in S_{\underline{v}}} J(\underline{x}_o, t_o, T; \underline{U}(\underline{x},t), \underline{v}) \tag{3.14}$$

By comparing (3.14) with (3.6) it is readily concluded that the solution to the differential game is also the solution to the min-max problem. It should be pointed out that the reverse is not necessarily true. That is, one may formulate a min-max problem which has a solution, while the corresponding differential game may have no solution.† However, if the scalars g and G in the performance index in (3.2) and the elements of the vector $\underline{f}$ in (3.1) are separable functions of $\underline{u}$ and $\underline{v}$, the corresponding min-max problem and differential game are equivalent [13].

In the following study of min-max problems for linear systems, we shall rely primarily on intuitive arguments to support our claims. Continued use is made of the Principle of Optimality [12]. In Appendix B, this principle is used to formally derive a partial differential equation (Issacs' "main equation" [13]) which will be used in the next section.

3.4 A Min-Max Problem with an Unconstrained Disturbance

It is well-known that the linear-plant quadratic performance criteria minimization problem, in which the square of the control appears in the performance index, is one of the few general minimization problems in which an optimal feedback control solution can readily be obtained. Since a min-max control must be feedback, it seems reasonable to first study the min-max analog of this minimization problem.

Let the system to be studied be described by the differential equation

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†For an example, see [29].

$$\underline{\dot{x}} = \underline{A}(t)\,\underline{x} + \underline{B}(t)\,\underline{u} + \underline{C}(t)\,\underline{v} \qquad t \in [t_o, T] \tag{3.15}$$

where $\underline{x}$ is a n-vector describing the state of the system, $\underline{u}$ is an m-vector of controls and $\underline{v}$ is an r-vector of disturbances acting on the system. $\underline{A}(t)$, $\underline{B}(t)$ and $\underline{C}(t)$ are piecewise-continuous time-varying matrices on $[t_o, T]$ with appropriate dimensions. It is assumed that $\underline{u} \in S_{\underline{u}}$ and $\underline{v} \in S_{\underline{v}}$ where $S_{\underline{u}}$ and $S_{\underline{v}}$ are subsets of the set of piecewise-continuous vector functions defined on $[t_o, T]$.

The performance criterion to be used is of the form

$$J(\underline{x}_o, t_o, T; \underline{u}, \underline{v}) = \frac{1}{2}\int_{t_o}^{T} [\underline{u}'\,\underline{D}_1\,\underline{u} - \underline{v}'\,\underline{D}_2\,\underline{v} + \underline{x}'\,\underline{Q}\,\underline{x}]$$

$$+ \frac{1}{2}\,\underline{x}'(T)\,\underline{M}\,\underline{x}(T) \tag{3.16}$$

where $\underline{D}_1$, $\underline{D}_2$, $\underline{Q}$ and $\underline{M}$ are constant, symmetric matrices with appropriate dimensions; $\underline{D}_1$ and $\underline{D}_2$ are positive-definite and $\underline{Q}$ and $\underline{M}$ are positive-semidefinite. The other aspects of this problem are as formulated in Section 3.2.

In the analogous minimization problem, the control is weighted in the performance index to insure the existence of a minimizing solution. Since in the min-max problem, the disturbance is presumed to maximize the performance index, it too must be weighted in the performance index if there is to be any hope for the existence of a min-max solution. The reason for the <u>negative</u> weighting is obvious since $\underline{v}$ maximizes.

A pursuit-evasion problem which is basically the same mathematical problem as the one formulated above with $\underline{Q} = 0$, has been studied in [17]. With some minor variation, the results which follow are essentially the same as those in the reference.

To study this problem it is convenient to use the min-max partial differential equation derived in Appendix B. Assuming that $F(\underline{x}(t),t)$ is the min-max return for a process starting at time $t \in [t_o, T]$ in state $\underline{x}(t)$, one may write

$$\min_{\underline{u} \in S_{\underline{u}}} \max_{\underline{v} \in S_{\underline{v}}} \left\{ \underline{F}'_x \underline{A} \underline{x} + \underline{F}'_x \underline{B} \underline{u} + \underline{F}'_x \underline{C} \underline{v} + \frac{1}{2} [\underline{x}' Q \underline{x} + \underline{u}' \underline{D}_1 \underline{u} - \underline{v}' \underline{D}_2 \underline{v}'] \right\} + F_t = 0 \tag{3.17}$$

Performing the indicated min-max operations one obtains

$$\left.\begin{aligned} \underline{u}^* &= - \underline{D}_1^{-1} \underline{B}' \underline{F}_x \\ \underline{v}^* &= \underline{D}_2^{-1} \underline{C}' \underline{F}_x \end{aligned}\right\} \tag{3.18}$$

The inverses of $\underline{D}_1$ and $\underline{D}_2$ exist since they are both positive definite. Substituting the expressions in (3.18) into (3.17), the resulting partial differential equation is

$$\underline{F}'_x \underline{A} \underline{x} + \frac{1}{2} \underline{x}' \underline{Q} \underline{x} - \frac{1}{2} \underline{F}'_x \underline{D} \underline{F}_x + F_t = 0 \tag{3.19}$$

where

$$\underline{D} = \underline{B} \underline{D}_1^{-1} \underline{B}' - \underline{C} \underline{D}_2^{-1} \underline{C}' \tag{3.20}$$

The boundary condition for (3.19) (see Appendix B) is

$$F(\underline{x}(T), T) \equiv \frac{1}{2} \underline{x}'(T) \underline{M} \underline{x}(T) \tag{3.21}$$

Consider as a possible form for $F(\underline{x}(t),t)$ the expression

$$F(\underline{x}(t),t) = \frac{1}{2} \underline{x}'(t) \underline{P}(t) \underline{x}(t); \qquad \underline{P}(t) = \underline{P}'(t) \tag{3.22}$$

From (3.21) it is clear that the boundary condition will be satisfied if

$$\underline{P}(T) = \underline{M} \tag{3.23}$$

If $\frac{1}{2} \underline{x}'(\tau) \underline{P}(\tau) \underline{x}(\tau)$ is the min-max return at time $\tau \in [t_o, T]$, then using (3.19) it is readily established that $\underline{P}(t)$ must satisfy

$$\dot{\underline{P}} = - \underline{A}'\underline{P} - \underline{P} \underline{A} + \underline{P}'\underline{D} \underline{P} - \underline{Q} \tag{3.24}$$

on $[\tau, T]$.

The expression in (3.24) is immediately recognized as a Riccati equation. Kalman [30,31] has pointed out that the solution to a Riccati equation on a finite time interval may not exist due to the phenomenon of <u>finite escape time</u>, i.e., some of the elements of $\underline{P}$ may go to infinity on the finite interval $[t_o,T]$.

If at some time $t_1 \epsilon [t_o,T)$ a finite escape occurs,† this means that the min-max return approaches infinity as τ approaches t_1 (assuming the return exists for $\tau > t_1$). In terms of the performance index, a finite escape simply means that the negative weighting of $\underline{v}$ is not sufficient to insure a bounded maximizing disturbance; i.e., a maximum with respect to $\underline{v}$ does not exist for $\tau \leq t_1$.

It is of interest to know under what conditions a finite escape will <u>not</u> occur. Kalman [30] has shown that the solution of (3.24) will exist on $[\tau,T]$ if the matrix $\underline{D}$ is positive-semidefinite on $[\tau,T]$. Though sufficient, this restriction on $\underline{D}$ is not always necessary as will be illustrated in an example below.

To work examples analytically, it is convenient to use the following transformation [30]. On the interval $(\tau,T]$ where $\underline{P}(t)$ exists, let

$$\underline{P}(T-t) = \underline{Z}(t)\, \underline{Y}^{-1}(t)$$

where $\underline{Z}(t)$ and $\underline{Y}(t)$ are nxn matrices satisfying

$$\left.\begin{array}{ll} \dot{\underline{Z}} = \underline{A}'\underline{Z} + \underline{Q}\,\underline{Y}\ ; & \underline{Z}(0) = \underline{M} \\ \dot{\underline{Y}} = \underline{D}\,\underline{Z} - \underline{A}\,\underline{Y}\ ; & \underline{Y}(0) = \underline{I}\ \text{(identity)} \end{array}\right\}$$

†The similarity between a point at which a finite escape occurs and a conjugate point (from the Jacobi condition in the calculus of variations [21]) has been noted in [17].

on $(\tau, T]$. The above equations are linear and for constant $\underline{A}$, $\underline{Q}$ and $\underline{D}$ may be solved using Laplace transforms. This approach has been used in obtaining the analytic solutions for Example 3.4.1.

Example 3.4.1

Consider the system

$$\dot{x}_1 = x_2 + u_1$$

$$\dot{x}_2 = \alpha\, u_2 + v$$

and the performance index

$$J = \frac{1}{2}\int_0^T [x_1^2 + u_1^2 + u_2^2 - \frac{1}{2} v^2]\, dt$$

Clearly

$$\underline{A} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} ; \ \underline{D} = \begin{pmatrix} 1 & 0 \\ 0 & \alpha^2-2 \end{pmatrix} ; \ \underline{Q} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

Consider first the solution for $\alpha = 3/2$. The matrix $\underline{D}$ is positive-definite for this value of α. The corresponding elements of the $\underline{P}$ matrix are

$$P_{11}(\lambda) = \frac{1}{\Delta(\lambda)}\left[\frac{\lambda}{2} + \frac{\sqrt{2}}{2}\left(\sinh \frac{\lambda}{\sqrt{2}}\right)\left(\cosh \frac{\lambda}{\sqrt{2}}\right) \right]$$

$$P_{12}(\lambda) = \frac{1}{\Delta(\lambda)}\left[\frac{\lambda^2}{4} + \sinh^2 \frac{\lambda}{\sqrt{2}} \right]$$

$$P_{22}(\lambda) = \frac{1}{\Delta(\lambda)}\left[\frac{2}{\sqrt{2}}\left(\sinh \frac{\lambda}{\sqrt{2}}\right)\left(\cosh \frac{\lambda}{\sqrt{2}}\right) - \lambda \right]$$

$$\Delta(\lambda) = \frac{\lambda^2}{8} + \frac{1}{4} + \frac{3}{4}\cosh^2 \frac{\lambda}{\sqrt{2}}$$

where $\lambda = T-t$. Inspection of $\Delta(\lambda)$ indicates that it is non-zero for all λ. Thus the elements of $\underline{P}$ remain finite for $t \in [t_o, T]$. Note that T may be infinite.

Now consider the solution for $\alpha = \sqrt{2}$. For this value of α, $\underline{D}$ is positive-semidefinite. The elements of the $\underline{P}$ matrix are

$$P_{11}(\lambda) = \frac{1}{\Delta(\lambda)} \sinh \lambda$$

$$P_{12}(\lambda) = \frac{1}{\Delta(\lambda)} \left[\cosh \lambda - 1 \right]$$

$$P_{22}(\lambda) = \frac{1}{\Delta(\lambda)} \left[\lambda \cosh \lambda - \sinh \lambda \right]$$

$$\Delta(\lambda) = \cosh \lambda$$

It is clear that $\Delta(\lambda) > 0$ for all λ. The elements of $\underline{P}$ will remain finite for all $t \in [t_o, T]$ provided $T < \infty$.

As a final example, consider the solution for the case $\alpha = 0$. For this case $\underline{D}$ is indefinite and there is a possibility of a finite escape. The elements of the $\underline{P}$ matrix are

$$P_{11}(\lambda) = \frac{1}{3\Delta(\lambda)} \left[(\cos \lambda)(\cosh \sqrt{2}\, \lambda) - \sqrt{2}(\sinh \sqrt{2}\, \lambda)(\cos \lambda) \right]$$

$$P_{12}(\lambda) = \frac{1}{9\Delta(\lambda)} \left[(\cos \lambda)(\cosh \sqrt{2}\, \lambda) + 2\sqrt{2}(\sin \lambda)(\sinh \sqrt{2}\, \lambda) - 1 \right]$$

$$P_{22}(\lambda) = \frac{1}{3\Delta(\lambda)} \left[(\sin \lambda)(\cosh \sqrt{2}\, \lambda) - \frac{1}{\sqrt{2}} (\cos \lambda)(\sinh \sqrt{2}\, \lambda) \right]$$

$$\Delta(\lambda) = \frac{1}{9} \left[4 + 5(\cos \lambda)(\cosh \sqrt{2}\, \lambda) + \sqrt{2}(\sin \lambda)(\sinh \sqrt{2}\, \lambda) \right]$$

The first positive zero of $\Delta(\lambda)$ occurs at roughly $\lambda_1 = 2.56$. Thus for $T < 2.56$, the min-max problem with $\alpha = 0$ has a solution. However, if $T > 2.56$, no solution exists due to the finite escape at $t_1 = T - 2.56$.

In addition to existence problems, it should be pointed out that there is another aspect of the problem formulated in this section which is undesirable from an engineering point of view. Note that if a solution exists, from (3.18) the worst case disturbance is

$$\underline{v}^* = \underline{D}_2^{-1}\, \underline{C}'\underline{P}\, \underline{x} \tag{3.25}$$

Suppose the system is at rest, i.e., $\underline{x}_o = 0$. The expression for $\underline{v}^*$ in (3.25) implies that the worst case disturbance will be zero. Thus if the system is at rest, the disturbance will find it more profitable to leave the system at rest than to disturb it. Though mathematically this is understandable, from a physical point of view it makes no sense at all. The reason why $\underline{v}^*$ is of the form shown in (3.25) is easily traced to the negative weighting of $\underline{v}$ in the performance index. Thus while on the one hand this weighting is necessary to keep the worst case disturbance bounded, on the other it leads to a meaningless result for $\underline{v}^*$.

If the disturbance is not weighted in the performance index, it must obviously be magnitude constrained by some other means. A logical way to do this is to <u>assume</u> that $\underline{v}$ is bounded at the outset.

Though this modification is both meaningful and intuitively appealing, it leads to a considerably more difficult type of min-max problem than the one considered in this section. This will become apparent from the analysis which follows.

3.5 <u>A Min-Max Problem with a Magnitude Constrained Disturbance</u>

For the problem studied in this section, consider the system described by

$$\dot{\underline{x}} = \underline{A}(t)\ \underline{x} + \underline{b}(t)\ u + \underline{c}(t)\ v \qquad t \in [t_o,T] \tag{3.26}$$

where u and v are both <u>scalars</u>, u being the control and v being a disturbance acting on the system. As in the last section, $\underline{x}$ is an n-vector describing the state of the system and $\underline{A}(t)$ is a piecewise-continuous nxn matrix defined on $[t_o,T]$. The vectors $\underline{b}(t)$ and $\underline{c}(t)$ are continuous on $[t_o,T]$. It is assumed that $u \in S_u$ and $v \in S_v$, where S_u is the set of all piecewise-continuous time functions on $[t_o,T)$ and S_v is the set of piecewise-continuous time functions satisfying

$$\left.\begin{aligned} \gamma_1(t) \leq v(t) \leq \gamma_2(t) \\ \gamma_1(t) < \gamma_2(t) \end{aligned}\right\} \tag{3.27}$$

The functions $\gamma_1(t)$ and $\gamma_2(t)$ are continuous on $[t_o,T]$.

Define the system error function

$$E(\underline{x}(t)) = |\underline{K}'\underline{x}(t)| \tag{3.28}$$

where $\underline{K}$ is a constant n-vector. Consider a performance index of the form

$$J(\underline{x}_o, t_o, T; u, v) = \int_{t_o}^{T} \frac{1}{2} u^2 dt + \frac{1}{2} E^2(\underline{x}(T)) \tag{3.29}$$

In all other respects, the min-max problem formulated here is the same as the general min-max problem described in Section 3.2.

A complete analysis of this problem is made here. In order to describe the results it will be necessary to define several pertinent time functions.

Let $\underline{R}(t)$ be an n-vector satisfying

$$\dot{\underline{R}}(t) = - \underline{A}'(t)\, \underline{R}(t) + \frac{1}{2} [\underline{b}'(t)\, \underline{R}(t)]^2\, \underline{R}(t); \quad \underline{R}(T) = \underline{K} \tag{3.30}$$

Next define the scalars $\theta(t)$ and $\phi(t)$ as solutions of

$$\dot{\theta}(t) = \frac{1}{2}(\underline{R}'(t)\, \underline{b}(t))^2\, \theta(t) - \frac{1}{2} [\gamma_2(t) + \gamma_1(t)]\, \underline{R}'(t)\, \underline{c}(t)\ ; \quad \theta(T) = 0 \tag{3.31}$$

and

$$\dot{\phi}(t) = \frac{1}{2}\, (\underline{R}'(t)\, \underline{b}(t))^2\, \phi(t) - \frac{1}{2} [\gamma_2(t) - \gamma_1(t)]\,|\underline{R}'(t)\, \underline{c}(t)|;\ \phi(T) = 0 \tag{3.32}$$

Finally, define the scalar $\xi(t)$ as the solution of

$$\dot{\xi}(t) = -\frac{1}{2}(\underline{R}'(t)\,\underline{b}(t))^2\,\xi(t) + \begin{cases} 0 & \text{if } \xi(t) = 0 \text{ and} \\ & \text{either } \delta(t) > 0 \text{ or} \\ & \underline{R}'(t)\,\underline{c}(t) = 0 \\ \delta(t) & \text{Otherwise} \end{cases} \qquad (3.33)$$

with $\xi(T) = 0$. The scalar $\delta(t)$ is defined as

$$\delta(t) = \frac{1}{2}|\underline{R}'(t)\,\underline{c}(t)|\,[\gamma_2(t) - \gamma_1(t)] - (\underline{R}'(t)\,\underline{b}(t))^2\,\phi(t) \qquad (3.34)$$

All of the above definitions hold on $[t_o,T]$. Note that $\xi(t)$ and $\phi(t)$ are non-negative on the problem interval. Note also that all of the above functions are independent of state.

Having defined the above functions, the results of the analysis may now be described.

Theorem 3.1

Let the general min-max problem be defined as in Section 3.2 and let (3.26) to (3.29) define the specific problem. The disturbance $v \in S_v$ is magnitude constrained. If for some $t \in [t_o,T]$ the state $\underline{x}(t)$ satisfies

$$|\underline{R}'(t)\,\underline{x}(t) + \theta(t)| \geq \xi(t) \qquad (3.35)$$

then the following statements hold:

1. The min-max return function $F(\underline{x}(t),t)$ which is defined by

$$F(\underline{x}(t),t) = \min_{u \in S_u}\ \max_{v \in S_v} J(\underline{x}(t), t, T; u, v)$$

is given by

$$F(\underline{x}(t),t) = \frac{1}{2}[|\underline{R}'(t)\,\underline{x}(t) + \theta(t)| + \phi(t)]^2 \qquad (3.36)$$

2. The min-max control u^* is given by

$$u^*(\underline{x}(t),t) = -\underline{R}'(t)\,\underline{b}(t)\,[\phi(t)\,\mathrm{sgn}(0,(\underline{R}'(t)\,\underline{x}(t) + \theta(t))) + \underline{R}'(t)\,\underline{x}(t) + \theta(t)] \qquad (3.37)$$

3. The worst case disturbance is of the form

$$v^*(\underline{x}(t),t) = \frac{1}{2}[\gamma_2(t) - \gamma_1(t)] \ \mathrm{sgn}(0,\underline{R}'(t)\ \underline{c}(t))$$

$$\mathrm{sgn}(q,\underline{R}'(t)\ \underline{x}(t) + \theta(t)) + \frac{1}{2}[\gamma_2(t) + \gamma_1(t)] \qquad (3.38)$$

where the constant q may be either 1 or -1.

The proof of this theorem is given in Appendix C.

Note that since q may be 1 or -1, v* is not unique. In view of the symmetry in the problem, this is not unexpected. It should be observed, however, that the min-max value of the performance index is unique, as is the control $u^*(\underline{x}(t),t)$.

If the results of Theorem 3.1 represents the min-max solution when

$$|\underline{R}'(t)\ \underline{x}(t) + \theta(t)| \geq \xi(t) \qquad (3.39)$$

what is the solution for those points in state space for which (3.39) does not hold? The answer, which is somewhat surprising, is that under such circumstances a min-max solution does not exist.

Theorem 3.2

For the min-max problem defined by (3.26) to (3.29), if for some state $\underline{x}(t)$, $t \in [t_o,T]$,

$$|\underline{R}'(t)\ \underline{x}(t) + \theta(t)| < \xi(t) \qquad (3.40)$$

then there is no solution to the problem.†

The proof of this theorem is presented in Appendix C.

On the surface, one would not expect to encounter any existence difficulties with this problem since v is bounded and the problem time $T-t_o$ is finite. Nevertheless, in regions of state space where (3.35)

† By no solution it is meant that a min-max control $u^*(\underline{x}(t),t)$ does not exist for all points in state space where (3.40) holds.

does not hold, min-max solutions do not exist. There does not appear to be any intuitively obvious reason why this is so. However, in studying the proof of Theorem 3.2, one comes to suspect that the existence problem is in part due to the fact that the control is weighted in the performance index. This suspicion is justified by the results of the next section in which the control is removed from the performance index and magnitude constrained.

The following example should help to illustrate the results of this section.

Example 3.5.1

Consider the system

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = u + v$$

where

$$|v| \le 1$$

The performance index is

$$J(\underline{x}_o, o, T; u, v) = \int_o^T \frac{u^2}{2}\, dt + \frac{1}{2}\, x_1^2(T)$$

From equation (3.30) it is noted that the elements of $\underline{R}$ satisfy

$$\dot{R}_1 = \frac{1}{2} R_2^2 R_1 \qquad\qquad R_1(T) = 1$$

$$\dot{R}_2 = - R_1 + \frac{1}{2} R_2^3 \qquad\qquad R_2(T) = 0$$

It can be easily verified that

$$\underline{R}(t) = \begin{pmatrix} R_1 \\ R_2 \end{pmatrix} = \frac{1}{[1 + \frac{1}{3}(T-t)^3]^{\frac{1}{2}}} \begin{pmatrix} 1 \\ T-t \end{pmatrix}$$

Because of the symmetric bounds on v

$$\theta(t) \equiv 0$$

Using (3.32)

$$\phi(t) = \frac{(T-t)^2}{2\,[1 + \frac{1}{3}(T-t)^3]^{\frac{1}{2}}}$$

With some effort, it may be determined that

$$\xi(t) = \begin{cases} 0 & \text{for } T-t < t_1 \\ [1 + \frac{1}{3}(T-t)^3]^{\frac{1}{2}} \left[t_1 - \dfrac{(T-t)^2}{2(1 + \frac{1}{3}(T-t)^3)} \right] & \text{for } T-t \geq t_1 \end{cases}$$

where

$$t_1 = \frac{1}{\sqrt[3]{6}}$$

This problem will have a min-max solution for all states $\underline{x}(t)$ if $T \leq t_1$ since $\xi(t) \equiv 0$ on $[o,t_1]$. If $T > t_1$, only for the states satisfying

$$|x_1(t) + (T-t)\,x_2(t)| \geq \left(t_1 - \frac{(T-t)^2}{2(1 + \frac{1}{3}(T-t)^3} \right)$$

will a min-max solution exist. Under these circumstances F, u* and v* are given by the expressions in Theorem 3.1. If the above inequality does not hold and $T > t_1$ there is no min-max solution to the problem.

In general, a control which minimizes the worst case value of the performance index in (3.29) cannot be found for all points in $(\underline{x},t)$ space. For those states lying in regions of state space where a min-max solution does not exist one would have to determine the control based on some other criteria. However, rather than do this, it seems more reasonable to consider a modification of the original problem: By removing the control from the performance index in (3.29) and appropriately bounding it, the existence difficulties associated with this problem disappear. This approach is taken in the next section.

3.6 A Min-Max Problem with Magnitude Constrained Control and Disturbance

In this section it is assumed that the governed by the vector differential equation

$$\dot{\underline{x}} = \underline{A}(t)\ \underline{x} + \underline{b}(t)\ u + \underline{c}(t)\ v \qquad t \in [t_o, T] \tag{3.41}$$

where $u \in S_u$ and $v \in S_v$. The symbols $\underline{x}$, $\underline{A}$, $\underline{b}$, $\underline{c}$, u, v and S_v have the same meanings as in the last section. The constraint set S_u is defined as the set of all piecewise-continuous functions on $[t_o, T]$ with magnitudes bounded by unity. For a performance index consider

$$J(\underline{x}_o,\ t_o,\ T;\ u,\ v) = \frac{1}{2}\, E^2(\underline{x}(T)) \tag{3.42}$$

where

$$E(\underline{x}(t)) = |\underline{K}'\underline{x}(t)| \tag{3.43}$$

In all other respects the problem to be studied in this section is the same as the general min-max problem formulated in Section 3.2.

In order to describe the solution to this problem, several time functions must first be defined.

Let $\underline{R}(t)$ be an n-vector satisfying

$$\dot{\underline{R}}(t) = -\ \underline{A}'(t)\ \underline{R}(t); \qquad \underline{R}(T) = \underline{K} \tag{3.44}$$

Define the scalars $\theta(t)$ and $\phi(t)$ as solutions of

$$\dot{\theta}(t) = -\ \frac{1}{2}\,[\gamma_2(t) + \gamma_1(t)]\ \underline{R}'(t)\ \underline{c}(t); \quad \theta(T) = 0 \tag{3.45}$$

and

$$\dot{\phi}(t) = -\ \frac{1}{2}\,[\gamma_2(t) - \gamma_1(t)]\,|\underline{R}'(t)\ \underline{c}(t)| + |\underline{R}'(t)\ \underline{b}(t)|\ ; \qquad \phi(T) = 0 \tag{3.46}$$

Finally, define the scalar $\xi(t)$ as the solution of

$$\dot{\xi}(t) = \begin{cases} 0 & \text{if } \xi(t) = 0 \text{ and } \\ & \delta(t) > 0 \\ \delta(t) & \text{Otherwise} \end{cases} \tag{3.47}$$

with $\xi(T) = 0$. The scalar $\delta(t)$ is defined as

$$\delta(t) = \frac{1}{2}\,[\gamma_2(t) - \gamma_1(t)]\,|\underline{R}'(t)\ \underline{c}(t)| - |\underline{R}'(t)\ \underline{b}(t)| \tag{3.48}$$

All of the above definitions hold on $[t_o, T]$.

The symbols $\underline{R}$, $\emptyset$, θ, ξ and δ have purposely been used in both this section and the last one to facilitate comparisons. It should be pointed out that these symbols represent <u>similar</u> quantities in the two sections, <u>not</u> the same quantities.

As a first step in describing the solution to the min-max problem in this section, the following theorem is presented:

<u>Theorem 3.3</u>

Let the general min-max problem be as defined in Section 3.2 and let (3.41) to (3.43) define the specific problem. Both $u \in S_u$ and $v \in S_v$ are magnitude constrained. If for some $t \in [t_o, T]$

$$|\underline{R}'(t)\ \underline{x}(t) + \theta(t)| \geq \xi(t) \tag{3.49}$$

then the following statements hold:

1. The min-max return function for the process is given by

$$F(\underline{x}(t),t) = \frac{1}{2}\,[\,|\underline{R}'(t)\ \underline{x}(t) + \theta(t)| + \emptyset(t)]^2 \tag{3.50}$$

2. The min-max control is

$$u^*(\underline{x}(t),t) = -\ \text{sgn}(o,\underline{R}'(t)\ \underline{b}(t)(\underline{R}'(t)\ \underline{x}(t) + \theta(t))) \tag{3.51}$$

3. The worst case disturbance is of the form

$$v^*(\underline{x}(t),t) = \frac{1}{2}\,[\gamma_2(t) - \gamma_1(t)]\ \text{sgn}(o,\underline{R}'(t)\ \underline{c}(t))$$
$$\text{sgn}(q,\underline{R}'(t)\ \underline{x}(t) + \theta(t)) + \frac{1}{2}\,[\gamma_2(t) + \gamma_1(t)] \tag{3.52}$$

where the constant q may be either 1 or -1.

The proof of this theorem may be found in Appendix C.

It is interesting to note the similarity between the results in the above theorem and those in Theorem 3.1. In both cases, the expressions for F and v* are of the same form.

The important difference between the two problems become apparent when the case in which (3.49) does not hold (i.e., $|\underline{R}'(t)\,\underline{x}(t) + \theta(t)| < \xi(t)$) is considered. In studying the analogous situation in the previous problem (see (3.40)), it was concluded that the min-max solution did not exist. Here a more palatable result has been obtained as evidenced by the following theorem.

<u>Theorem 3.4</u>

At all points in $(\underline{x},t)$ space satisfying

$$|\underline{R}(t)\,\underline{x}(t) + \theta(t)| < \xi(t)$$

the min-max return $F(\underline{x}(t),t)$ is given by

$$F(\underline{x}(t),t) = \frac{1}{2}\,\phi^2(\hat{t}) \tag{3.53}$$

where $\hat{t}$ is the first zero of $\xi(\tau)$ on the interval $[t,T]$. So long as the above inequality holds, the choice of $u \in S_u$ and $v \in S_v$ will have no effect on the value of the min-max return.

A proof of this theorem is given in Appendix C.

Theorem 3.4 indicates that in a region in state space, the choice of control has no effect on the worst case value of the system performance index. When the state lies in this region, one might consider choosing the control based on some secondary performance index.

The following examples are presented to illustrate the results of this section.

Example 3.6.1

Consider the system

$$\dot{x}_1 = x_2 - v$$

$$\dot{x}_2 = -25\,x_1 - 2\,x_2 + v + 2\,u$$

where

$$|u| \leq 1$$

$$|v| \leq 1$$

Let the performance index be

$$J(\underline{x}_o, 0, 3; u, v) = \frac{1}{2}\,x_1^2(3)$$

The control u is given by (3.51) whenever $|\underline{R}(t)\,\underline{x}(t) + \theta(t)| \geq \xi(t)$ and is chosen to be zero otherwise. Data for a trajectory starting at $\underline{x}_o = 0$ and resulting from the disturbance forcing function v given by (3.52), is shown in Figure 3.1.

The magnitude of $x_1(3)$ is the worst case value; i.e., there is no $v \in S_v$ which can result in a larger value for $|x_1(3)|$. In addition, there is no admissable controller which will result in a smaller worst case value.

Example 3.6.2

It is interesting to consider the same problem as above with the roles of the control and the disturbance interchanged. Specifically, let

$$\dot{x}_1 = x_2 - u$$

$$\dot{x}_2 = -25\,x_1 - 2\,x_2 + u + 2\,v$$

$$|u| \leq 1$$

$$|v| \leq 1$$

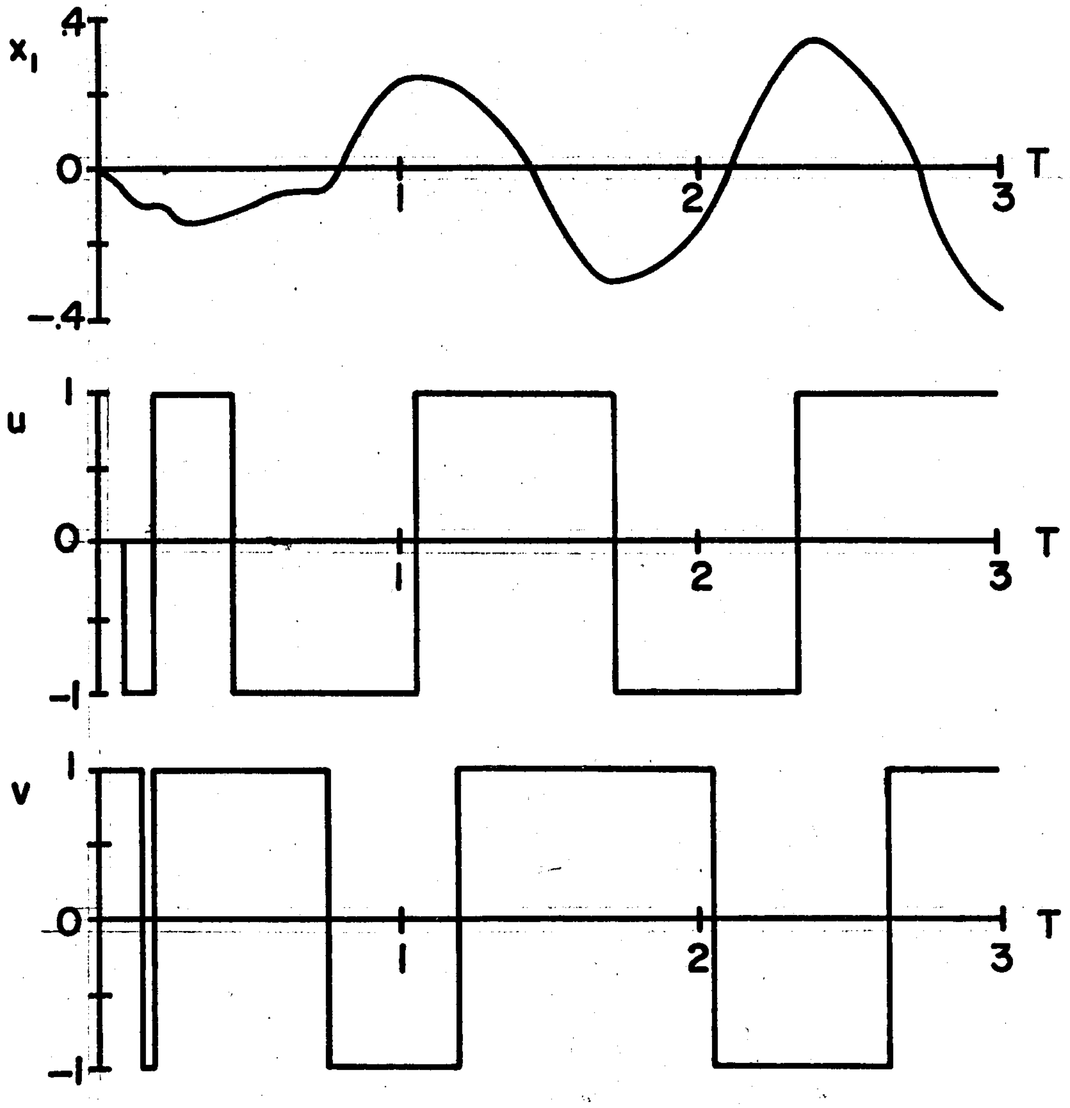

Figure 3.1. Trajectory data for Example 3.6.1.

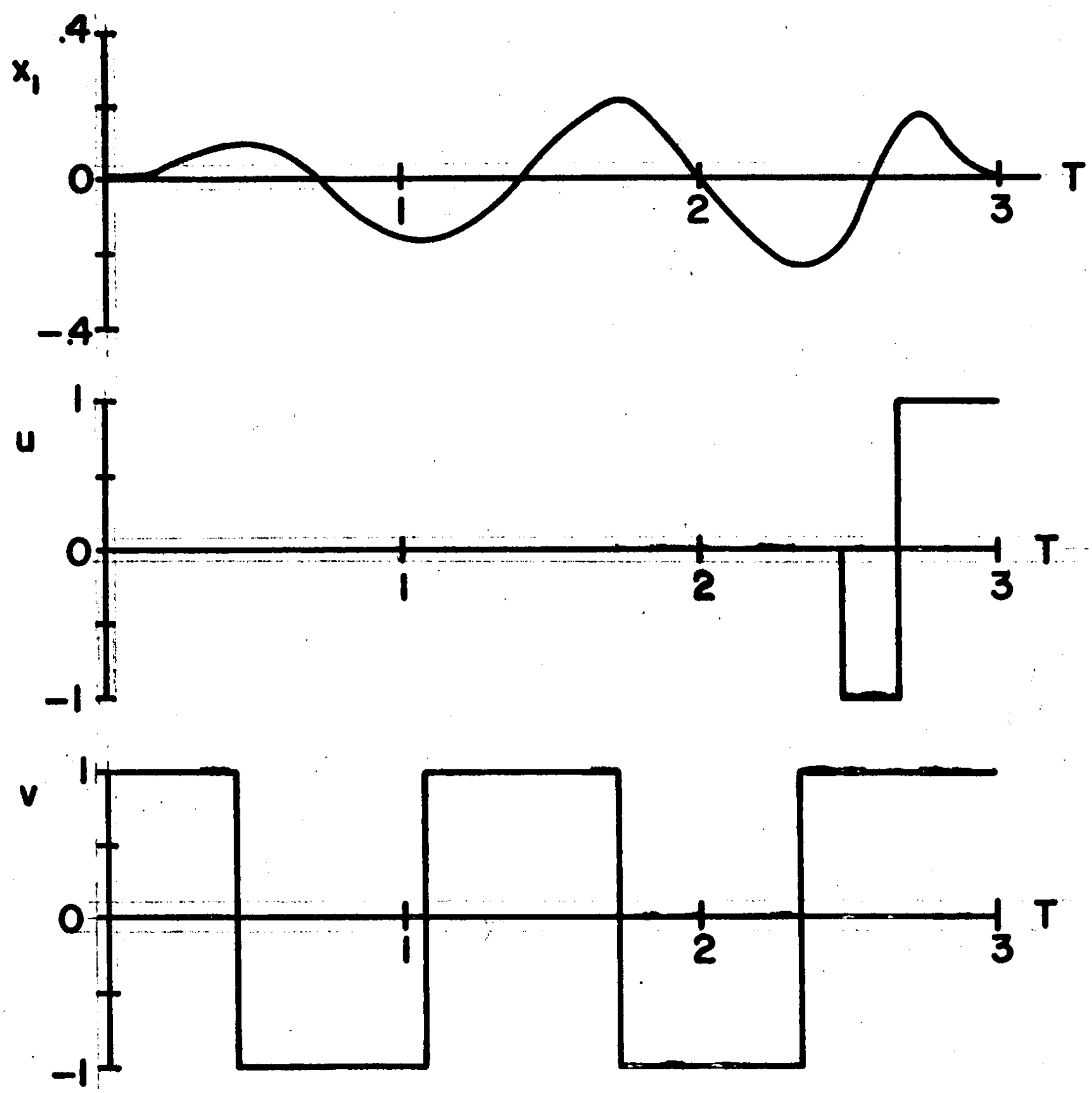

Figure 3.2. Trajectory data for Example 3.6.2.

and

$$J(\underline{x}_o, 0, 3; u, v) = \frac{1}{2} x_1^2(3)$$

As in the previous example, u is given by (3.51) for $|\underline{R}(t)\ \underline{x}(t) + \theta(t)| \geq \xi(t)$ and zero otherwise. The initial state is still zero and v is given by (3.52) for the entire trajectory. The resulting curves for x_1, u and v are shown in Figure 3.2.

It is immediately observed that the final value of $x_1(t)$ is zero. This means that for <u>any</u> disturbance bounded by unity, the controller u* will drive $x_1(T)$ to zero! Clearly no other controller can do better.

3.7 Summary

The material presented in this chapter represents the results of an investigation of min-max problems for linear systems. Three problems have been studied, the major differences among them being in the form of the performance index used in each.

In the first problem the performance index contains a positively weighted control and a negatively weighted disturbance. It has been noted that the min-max solution to this problem does not always exist due to the phenomenon of finite escape time. A more serious criticism of the problem is that it leads to a form for the maximizing disturbance (a linear function of state) which makes little sense.

In an attempt to avoid existence difficulties and to obtain more meaningful results, in the next problem studied the magnitude of the disturbance is constrained. A performance index containing the integral of the square of the control plus the square of the terminal value of a system error is used. A surprising result is that the solution to this min-max problem does not exist for all points in state space.

In the third problem studied, both the control and the disturbance are magnitude constrained. The performance index is the square of the terminal value of a system error. A complete solution to this problem has been presented. It was shown that in some regions of state space, the choice of control has no effect on the min-max value of the performance index.

CHAPTER 4

CONCLUSION

4.1 Summary and Conclusions

In the preceding chapters, two problems arising from the worst case disturbance approach have been investigated. In the first of these, the error analysis problem of Chapter 2, the object has been to determine the worst case (maximum) value of a system performance index (a quadratic function of state evaluated at a fixed time) due to a bounded disturbance acting on a general linear system. The approach taken in studying this problem has been to relate this general performance index to a special one for which the corresponding optimal (worst case) return function could be found. This has led to what is undoubtedly the major contribution of this work. It has been shown (Theorem 2.3) that the general error analysis problem can be solved by maximizing the return for the special problem with respect to the components of a vector of unit length.

In addition to simplifying the problem and providing further insight into its nature through a geometric interpretation, this contribution has led to several other results worthy of mention. First, the existence of a maximizing solution is clearly established. Second, the question of what the solution is on a singular surface is completely answered. Third, the relationship between a local maximum of the return function F and a solution satisfying the necessary conditions of the Maximum Principle (Theorem 2.1 and 2.2) is made clear.

From the results presented in Chapter 2, it has been noted that F is a multimodal function. To determine the global maximum of this function with a reasonable amount of computation, an efficient local hill-climbing technique is required. By recognizing that the return F is a convex function, it has been possible to develop an algorithm in which convergence to a local maximum is guaranteed. It has been noted that the cost of computation per iteration is small because no integration is required.

Chapter 3 has been devoted to a study of the min-max problem. The problem has been to determine a feedback controller having associated with it the smallest worst case value of a prescribed system performance index. The chapter contains the results of an investigation of min-max problems involving linear systems and three different types of performance indices. For the first of these problems, it has been noted that a solution might not exist. In addition, it has been pointed out that the problem leads to a meaningless form for the worst case disturbance. Unexpectedly, existence difficulties also occurred in the second problem for states lying in certain regions of $(\underline{x},t)$ space. For the third problem studied, a complete min-max solution has been presented.

A survey of the literature indicates that there are a significant number of results for differential games at the two ends of the research spectrum. At one end there is an elaborate general theory, as evidenced by the work reported in [14-16]. Although this theory is of great value in providing the basic tools needed to study differential games, it does not lead to any simple method for actually solving problems.

At the other end of the spectrum, there are a number of specific numerical examples which have been solved [13,20]. Though these examples are most helpful in providing insight into the nature of the differential game, they are too specific to be of general use to the control engineer.

The results reported in Chapter 3 lie somewhere between these two extremes. Emphasis has been placed on the problem of determining a min-max controller for a general linear system. Since this problem is quite difficult and since the literature contains few results at this level, any definitive statements about the solutions of min-max problems are considered to be worthwhile contributions. Theorems 3.1 to 3.4 are cases in point. These theorems should prove useful in furthering the development of a general min-max theory for linear systems.

4.2 Recommendations for Further Study

Since the proof of Theorem 2.3 in Chapter 2 does not depend on the assumption that the plant being analyzed is linear, the theorem also holds for nonlinear systems. Of course, to make use of the theorem one must find the return function $F(\underline{x}_o, t_o, \underline{\alpha})$. Although this is a formidable problem for a general nonlinear system, it is conceivable that for special classes of systems (i.e., possibly systems with a single nonlinearity of a particular type) this return can be determined. Since the extension of the results of Chapter 2 to nonlinear systems would be of significant value, this problem is one which further research should be undertaken.

A second problem worthy of investigation can be described as follows. Consider the functionals

$$J(\underline{x}_o, t_o, T; u, v) = \frac{1}{2} \underline{x}'(T) \underline{Q} \underline{x}'(T) \tag{4.1}$$

and

$$\overline{J}(\underline{x}_o, t_o, T; u, v) = \frac{1}{2} [\underline{x}'(T) \underline{M}^{-1} \underline{\alpha}]^2 \tag{4.2}$$

where $\underline{\alpha} \in \Omega^r$ (see Chapter 2 for definitions of $\underline{\alpha}$, $\underline{Q}$, $\underline{M}$ and Ω^r) and $\underline{x}(t)$ is the state of the system described by equation (3.41) in Section 3.6. Theorems 3.3 and 3.4 may be used to determine the min-max return

$$F(\underline{x}_o, t_o, \underline{\alpha}) = \min_{u \in S_u} \max_{v \in S_v} \overline{J}(\underline{x}_o, t_o, T; u, v)$$

It is clear from Theorem 2.3 that

$$\min_{u \in S_u} \max_{v \in S_v} J(\underline{x}_o, t_o, T; u, v) = \max_{\underline{\alpha} \in \Omega^r} F(\underline{x}_o, t_o, \underline{\alpha}) \tag{4.3}$$

when $\underline{b}(t) \equiv 0$, since the resulting problem involves only maximization with respect to v. Using arguments similar to those used in the proof of Theorem 2.3, one may show that (4.3) also holds for the case $\underline{b}(t) \not\equiv 0$, $\underline{c}(t) \equiv 0$. The obvious question is whether (4.3) holds when neither $\underline{b}(t)$ nor $\underline{c}(t)$ are zero; i.e., for the min-max problem. Since an answer to this question would represent a significant contribution to the general theory of min-max problems for linear systems, it is suggested as a subject for further study.

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APPENDIX A

APPENDIX A

PROPERTIES OF $\underline{\lambda}(t,\underline{\alpha})$

In this appendix, it will be shown that the vector $\underline{\lambda}(t,\underline{\alpha})$ defined in Chapter 2 is continuous on $[t_o,T]$ and satisfies the adjoint equation of the Maximum Principle.

A.1 A Continuous Time-Function

In this section it will be shown that $\underline{\lambda}(t,\underline{\alpha})$ is continuous on $[t_o,T]$. Using (2.35), we may write

$$\lambda(t,\underline{\alpha}) = \underline{P}(t)\ \underline{\alpha}[\,|y(t)| + \phi(t,\alpha)]\ \mathrm{sgn}[1,\dot{y}(t)] \tag{A.1}$$

where

$$y(t) = \underline{x}'(t,\underline{\alpha})\ \underline{P}(t)\ \underline{\alpha} + \theta(t,\underline{\alpha}) \tag{A.2}$$

Since $\underline{P}(t)$, $\underline{\alpha}$, $y(t)$ and $\phi(t,\underline{\alpha})$ are all continuous on $[t_o,T]$, to show that $\underline{\lambda}(t,\underline{\alpha})$ is continuous it will be sufficient to show that either

$$y(t) \geq 0 \tag{A.3}$$

on $[t_o,T]$, or

$$y(t) \leq 0 \tag{A.4}$$

on $[t_o,T]$. Let us consider the time-derivative of $y(t)$. After some manipulation, it may be determined that

$$\dot{y}(t) = \frac{1}{2}\,[\gamma_2(t) - \gamma_1(t)]\ |\underline{c}'(t)\ \underline{P}(t)\ \underline{\alpha}|\,\mathrm{sgn}(1,y(t)) \tag{A.5}$$

It follows from this expression that

$$\dot{y}(t) \geq 0 \quad \text{whenever} \quad y(t) \geq 0$$

and

$$\dot{y}(t) < 0 \quad \text{whenever} \quad y(t) < 0$$

These inequalities imply that $y(t)$ must satisfy either (A.3) or (A.4) Thus $\underline{\lambda}(t,\underline{\alpha})$ is continuous on $[t_o,T]$.

A.2 <u>A Solution of the Maximum Principle Equations</u>

In view of the discussion in the last section, it may be easily established that $\underline{\lambda}(t,\underline{\alpha})$ is differentiable on $(t_o,T]$. Thus we proceed to write

$$\begin{aligned}\dot{\underline{\lambda}}(t,\underline{\alpha}) &= \dot{\underline{P}}(t)\ \underline{\alpha}[|y(t)| + \phi(t,\underline{\alpha})]\ \mathrm{sgn}[1,y(t)] \\ &\quad + \underline{P}(t)\ \underline{\alpha}[\dot{y}(t)\ \mathrm{sgn}[1,y(t)] + \dot{\phi}(t,\underline{\alpha}]\ \mathrm{sgn}[1,y(t)] \\ &= -\ \underline{A}'(t)\ \underline{P}(t)\ \underline{\alpha}[|y(t)| + \phi(t,\underline{\alpha})]\ \mathrm{sgn}[1,y(t)] \\ &\quad + \underline{P}(t)\ \underline{\alpha}[\tfrac{1}{2}\ (\gamma_2(t) - \gamma_1(t)]\,|c'(t)\ \underline{P}(t)\ \underline{\alpha}| + \dot{\phi}(t,\underline{\alpha})]\end{aligned} \tag{A.6}$$

But from (2.33) it is observed that the second term in the above expression is zero. Using this fact and equation (A.1) it is clear that

$$\dot{\underline{\lambda}}(t,\underline{\alpha}) = -\ \underline{A}'(t)\ \underline{\lambda}(t,\underline{\alpha}) \tag{A.7}$$

Thus $\underline{\lambda}(t,\underline{\alpha})$ satisfies the adjoint equation of the Maximum Principle.

APPENDIX B

APPENDIX B

THE MIN-MAX PARTIAL DIFFERENTIAL EQUATION

In this appendix, a partial differention equation similar to the classical Hamilton-Jacobi equation of the calculus of variations [21] is derived. The result obtained is not new (i.e., see [13]) and the derivation is not rigorous. Nevertheless, a derivation based on the Principle of Optimality [12] is intuitively appealing and will therefore add further to the understanding of the min-max problem.

The development starts with the definition of a min-max return function. With reference to the problem formulated in Section 3.2, define $F(\underline{x}(t),t)$ as the min-max return from a process starting at time $t \in [t_o,T]$ in state $x(t)$. That is

$$F(x(t),t) = \min_{\substack{\underline{u}(\tau) \\ \tau \in [t,T]}} \max_{\underline{v}(\tau)} \left[\int_t^T g(\underline{x}, t, \underline{u}, \underline{v})\, dt + G(\underline{x}(T),T)\right] \tag{B.1}$$

It is understood that $\underline{u} \in S_{\underline{u}}$ and $\underline{v} \in S_{\underline{v}}$. We shall assume that $\dot{F}$, $\ddot{F}$, g and $\dot{g}$ are piecewise-continuous time functions along a min-max trajectory.

The expression for $F(\underline{x}(t),t)$ above, may also be written as

$$F(\underline{x}(t),t) = \min_{\substack{\underline{u}(\tau) \\ \tau \in [t,T]}} \max_{\underline{v}(\tau)} [I_1 + I_2] \tag{B.2}$$

where

$$I_1 = I_1(\underline{x}(t), t, t + \Delta; \underline{u}(\tau), \underline{v}(\tau); \tau \in [t,t + \Delta])$$

$$= \int_t^{t + \Delta} g(\underline{x}, t, \underline{u}, \underline{v})\, dt \tag{B.3}$$

and

$$I_2 = I_2(\underline{x}(t + \Delta), t + \Delta, T; \underline{u}(\tau), \underline{v}(\tau); \tau \in [t + \Delta, T])$$

$$= \int_{t + \Delta}^{T} g(\underline{x}, t, \underline{u}, \underline{v})\, dt + G(\underline{x}(T), T) \qquad (B.4)$$

Since I_1 is independent of $v(\tau)$ for $\tau \in [t + \Delta, T]$ we may write

$$F(\underline{x}(t),t) = \min_{\substack{\underline{u}(\tau) \\ \tau \in [t,T]}} \max_{\substack{\underline{v}(\tau_1) \\ \tau_1 \in [t,t + \Delta]}} [I_1 + \max_{\substack{\underline{v}(\tau_2) \\ \tau_2 \in [t + \Delta, T]}} I_2] \qquad (B.5)$$

Now consider $\underline{u}(\tau_2)$, $\tau_2 \in [t + \Delta, T]$. As stipulated in the formulation of the problem, $\underline{u}(\tau_2)$ is to be a function of $\underline{x}(\tau_2)$. This means that $\underline{u}(\tau_2)$ has knowledge of the past history of $\underline{v}(\tau_1)$, $\tau_1 \in [t, t + \Delta]$. Hence the min operation for $\underline{u}(\tau_2)$ and the max operation for $\underline{v}(\tau_1)$ may be interchanged. The resulting expression is

$$F(\underline{x}(t),t) = \min_{\substack{\underline{u}(\tau_1) \\ \tau_1 \in [t,t + \Delta]}} \max_{\underline{v}(\tau_1)} \min_{\substack{\underline{u}(\tau_2) \\ \tau_2 \in [t + \Delta, T]}} [I_1 + \max_{\substack{\underline{v}(\tau_2) \\ \tau_2 \in [t + \Delta, T]}} I_2] \qquad (B.6)$$

But since I_1 doesn't depend upon $\underline{u}(\tau_2)$ we may write

$$F(\underline{x}(t),t) = \min_{\substack{\underline{u}(\tau_1) \\ \tau_1 \in [t,t + \Delta]}} \max_{\underline{v}(\tau_1)} [I_1 + \min_{\substack{\underline{u}(\tau_2) \\ \tau_2 \in [t + \Delta, T]}} \max_{\underline{v}(\tau_2)} I_2] \qquad (B.7)$$

The expression in (B.7) may be taken as a statement of the Principle of Optimality [12] for the min-max problem.

From (B.2) through (B.4) and (B.7) it is clear that

$$F(\underline{x}(t),t) = \min_{\substack{\underline{u}(\tau_1) \\ \tau_1 \in [t,t + \Delta]}} \max_{\underline{v}(\tau_1)} [\int_t^{t + \Delta} g(\underline{x}, t, \underline{u}, \underline{v})\, dt + F(\underline{x}(t + \Delta), t + \Delta)] \qquad (B.8)$$

Since $\dot{g}$, $\dot{F}$ and $\ddot{F}$ are piecewise-continuous we may expand the right side of (B.8) in a Taylor series. Thus

$$F(\underline{x}(t),t) = \min_{\substack{\underline{u}(\tau_1) \\ \tau_1 \epsilon [t,t+\Delta]}} \max_{\underline{v}(\tau_1)} [\Delta\, g(\underline{x}, t, \underline{u}, \underline{v}) + F(\underline{x}(t),t) + \Delta\, \dot{F}(\underline{x}(t),t) + 0_{\Delta}^{2}] \tag{B.9}$$

where

$$\lim_{\Delta \to 0} \frac{0_{\Delta}^{2}}{\Delta} = 0 \tag{B.10}$$

We may now write

$$F(\underline{x}(t),t) = F(\underline{x}(t),t) + \min_{\substack{\underline{u}(\tau_1) \\ \tau_1 \epsilon [t,t+\Delta]}} \max_{\underline{v}(\tau_1)} [\Delta\, g(\underline{x}, t, \underline{u}, \underline{v}) + \Delta\, \dot{F}(\underline{x}(t),t) + 0_{\Delta}^{2}] \tag{B.11}$$

Therefore

$$\min_{\substack{\underline{u}(\tau_1) \\ \tau_1 \epsilon [t,t+\Delta]}} \max_{\underline{v}(\tau_1)} [\Delta\, g(\underline{x}, t, \underline{u}, \underline{v}) + \Delta\, \dot{F}(\underline{x}(t),t) + 0_{\Delta}^{2}] = 0 \tag{B.12}$$

Dividing by Δ and then letting Δ go to zero

$$\min_{\underline{u}(t)} \max_{\underline{v}(t)} [g(\underline{x}, t, \underline{u}, \underline{v}) + \dot{F}(\underline{x}(t),t)] = 0 \tag{B.13}$$

Noting that

$$\begin{aligned} \dot{F}(\underline{x}(t),t) &= \underline{F}_x'(\underline{x}(t),t)\, \dot{\underline{x}} + F_t(\underline{x}(t),t) \\ &= \underline{F}_x'\, \underline{f}(\underline{x}, t, \underline{u}, \underline{v}) + F_t \end{aligned} \tag{B.14}$$

the expression in (B.13) may now be written as

$$\min_{\underline{u}(t)} \max_{\underline{v}(t)} [g(\underline{x}, t, \underline{u}, \underline{v}) + \underline{F}_x'\, \underline{f}(\underline{x}, t, \underline{u}, \underline{v}) + F_t] = 0 \tag{B.15}$$

We shall refer to (B.15) as the min-max partial differential equation.

Using (B.1), it is noted that

$$F(\underline{x}(T),T) \equiv G(\underline{x}(T),T) \tag{B.16}$$

and thus

$$\underline{F}_x(\underline{x}(T),T) = \underline{G}_x(\underline{x}(T),T)) \tag{B.17}$$

The expression in (B.17) is a boundary condition for the min-max partial differential equation.

APPENDIX C

APPENDIX C

PROOFS OF THEOREMS 3.1 to 3.4

In this appendix, the proofs of Theorems 3.1 through 3.4 are presented. The notation used here is consistent with that used in Section 3.5 and 3.6 of Chapter 3.

C.1 Proof of Theorem 3.1

At least two different approaches may be used to prove Theorem 3.1. The first would involve showing that the return $F(\underline{x}(t),t)$ defined in (3.36) does indeed satisfy the min-max partial differential equation presented in Appendix B. Statements 2 and 3 of the theorem would then follow directly. In the proof that follows, a somewhat different approach is used because it is felt that it will lead to a better understanding of the problem.

Let us start by making several simplifying transformations. First define the scalar y(t) as

$$y(t) = \beta(t)\,[\underline{R}'(t)\,\underline{x}(t) + \theta(t)] \qquad t \in [t_o,T] \tag{C.1}$$

where $\underline{R}(t)$ and $\theta(t)$ are defined in (3.30) and (3.31) and $\beta(t)$ is a scalar satisfying

$$\beta(t) = e^{\left[\int_t^T \frac{(\underline{R}'(\tau)\,\underline{b}(\tau))^2}{2} d\tau\right]} \qquad t \in [t_o,T] \tag{C.2}$$

Next define the scalars $\bar{u}$ and $\bar{v}$ as

$$\bar{u} = u + \underline{R}'(t)\, \underline{b}(t)(\underline{R}'(t)\, \underline{x} + \theta(t)) = u + \frac{\underline{R}'(t)\, \underline{b}(t)}{\beta(t)}\, y \tag{C.3}$$

and

$$\bar{v} = \left[v - \frac{1}{2}(\gamma_2(t) + \gamma_1(t)) \right] \left[\frac{2}{\gamma_2(t) - \gamma_1(t)} \right] \tag{C.4}$$

on $[t_o, T]$. Let $S_{\bar{v}}$ be the set of all piecewise-continuous time functions on $[t_o, T]$ such that if $\bar{v} \in S_{\bar{v}}$ then

$$|\bar{v}(t)| \leq 1 \tag{C.5}$$

Note that $v \in S_v \Leftrightarrow \bar{v} \in S_{\bar{v}}$. A similar statement holds for u and $\bar{u}$.

Let

$$\underline{d}(t) = \frac{\gamma_2(t) - \gamma_1(t)}{2}\, \underline{c}(t) \qquad t \in [t_o, T] \tag{C.6}$$

It is easily verified by differentiating the expression in (C.1) that

$$\dot{y} = \beta(t)\, \underline{R}'(t)\, \underline{b}(t)\, \bar{u} + \beta(t)\, \underline{R}'(t)\, \underline{c}(t)\, \bar{v} \tag{C.7}$$

For convenience we shall refer to $\bar{u}$ and $\bar{v}$ as a "control" and a "disturbance" respectively. If we next define the functional

$$\bar{J}(y(t_o), t_o, T; \bar{u}, \bar{v}) = J(\underline{x}(t_o), t_o, T; u, v) \tag{C.8}$$

then the performance index expressed in (3.29) may be written as

$$\bar{J}(y(t_o), t_o\ T; \bar{u}, \bar{v}) = \frac{1}{2} \int_{t_o}^{T} \left(\bar{u} - \frac{\underline{R}'(t)\, \underline{b}(t)}{\beta(t)}\, y \right)^2 dt + \frac{y^2(T)}{2} \tag{C.9}$$

Since

$$\min_{\bar{u} \in S_{\bar{u}}} \max_{\bar{v} \in S_{\bar{v}}} \bar{J}(y(t_o), t_o, T; \bar{u}, \bar{v})$$

$$= \min_{u \in S_u} \max_{v \in S_v} J(\underline{x}(t_o), t_o, T; u, v) \tag{C.10}$$

it will be sufficient to consider

$$\min_{\bar{u} \in S_{\bar{u}}} \max_{\bar{v} \in S_{\bar{v}}} \bar{J}(y(t_o), t_o, T; \bar{u}, \bar{v})$$

The minimizing control $\bar{u}^*$, and the maximizing disturbance $\bar{v}^*$ may be determined for $t \in [\tau, T] \subset [t_o, T]$ by considering the return

$$\bar{F}(y(\tau), \tau) = \min_{\bar{u} \in S_{\bar{u}}} \max_{\bar{v} \in S_{\bar{v}}} \bar{J}(y(\tau), \tau, T; \bar{u}, \bar{v})$$

$$= \min_{\bar{u} \in S_{\bar{u}}} \max_{\bar{v} \in S_{\bar{v}}} \left[\frac{1}{2} \int_{\tau}^{T} \left(\bar{u} - \frac{\underline{R}'(t)\, \underline{b}(t)}{\beta(t)}\, y \right)^2 dt + \frac{y^2(T)}{2} \right] \tag{C.11}$$

This follows from the Principle of Optimality [12]. Now consider the functional

$$I(y(\tau), \tau, T; \bar{u}, \bar{v})$$

$$= \frac{y^2(\tau)}{2\beta^2(\tau)} + \int_{\tau}^{T} \left[\frac{\bar{u}^2}{2} + \frac{\underline{R}'(t)\, \underline{d}(t)}{\beta(t)}\, y\bar{v} \right] dt \tag{C.12}$$

Note that for $\tau = T$

$$I(y(T), T, T; \bar{u}, \bar{v}) = \bar{J}(y(T), T, T; \bar{u}, \bar{v}) \tag{C.13}$$

Furthermore, by differentiating $\bar{J}(y(\tau), \tau, T; \bar{u}, \bar{v})$ and $I(y(\tau), \tau, T; \bar{u}, \bar{v})$, it may easily be verified that

$$\dot{I}(y(t), t, T; \bar{u}, \bar{v}) = \dot{\bar{J}}(y(t), t, T; \bar{u}, \bar{v}) \tag{C.14}$$

at all points on $[t_o, T]$. Thus it must follow that

$$\bar{J}(y(\tau), \tau, T; \bar{u}, \bar{v}) \equiv I(y(\tau), \tau, T; \bar{u}, \bar{v}) \tag{C.15}$$

for $\tau \in [t_o, T]$. Thus

$$\bar{F}(y(\tau), \tau) = \frac{y^2(\tau)}{2\beta^2(\tau)}$$

$$+ \min_{\bar{u} \in S_{\bar{u}}} \max_{\bar{v} \in S_{\bar{v}}} \left[\int_{\tau}^{T} \left(\frac{\bar{u}^2}{2} + \frac{\underline{R}'(t)\, \underline{d}(t)}{\beta(t)}\, y\bar{v} \right) dt \right] \tag{C.16}$$

Assume now that the minimizing control $\bar{u}^*$ is in a set of controls $S^*_{\bar{u}} \subset S_{\bar{u}}$ having the following properties:

$$\left.\begin{array}{ll} 1. & \bar{u}(y,t) = - \bar{u}(-y,t) \\ 2. & |\bar{u}(y_1,t)| \geq |\bar{u}(y_2,t)| \quad \text{if} \quad |y_1| \geq |y_2| \end{array}\right\} \tag{C.17}$$

on $[t_o,T]$. This assumption will be justified shortly.

Let $z(t,y(\tau))$ represents a solution of (C.7) on $[\tau,T]$ using any $\bar{u} \in S^*_{\bar{u}}$ and any $\bar{v} \in S_{\bar{v}}$. The initial point of this trajectory is taken as $z(\tau,y(\tau)) = y(\tau)$. Let $z^*(t,y(\tau))$ be a solution of (C.7) using $\bar{v} = \bar{v}^*$ and any $\bar{u} \in S^*_{\bar{u}}$. We can now make the following statement.

Statement C.1.1

If $\bar{u} \in S^*_{\bar{u}}$, then

$$\max_{\bar{v} \in S_{\bar{v}}} \int_\tau^T \left[\frac{\bar{u}^2(z,t)}{2} + \frac{\underline{R}'(t)\ \underline{d}(t)}{\beta(t)}\ z\bar{v} \right] dt$$

$$= \int_\tau^T \left[\frac{\bar{u}^2(z^*,t)}{2} + \left| \frac{R'(t)\ d(t)}{\beta(t)}\ z^* \right| \right] dt \tag{C.18}$$

and

$$\bar{v}^* = \operatorname{sgn}\left(0, \frac{\underline{R}'(t)\ \underline{d}(t)}{\beta(t)} \right) \operatorname{sgn}(q,z^*) \ ; \qquad q = \pm 1 \tag{C.19}$$

Note that $\bar{v}^*$ is not unique since q may be 1 or -1. To prove this statement we first note that

$$|z^*(t,y(\tau))| \geq |z(t,y(\tau))| \tag{C.20}$$

for all $t \in [\tau,T]$. This follows directly from Property 1 in (C.17) and an inspection of (C.7). It is clear from (C.20) and Property 2 in (C.17) that

$$\max_{\bar{v} \in S_{\bar{v}}} \int_\tau^T \frac{\bar{u}^2(z,t)}{2}\ dt = \int_\tau^T \frac{\bar{u}^2(z^*,t)}{2}\ dt \tag{C.21}$$

for $\bar{u} \in S^*_{\bar{u}}$.

Again using (C.20) and noting the form of $\bar{v}^*$ in (C.19), it is seen that

$$\max_{\bar{v} \in S_{\bar{v}}} \int_\tau^T \left(\frac{(\underline{R}'(t)\ \underline{d}(t))}{\beta(t)}\ z\bar{v} \right) dt = \int_\tau^T \left| \frac{\underline{R}'(t)\ \underline{d}(t)}{\beta(t)}\ z^* \right| dt \tag{C.22}$$

The results expressed in (C.18) and (C.19) follow directly from (C.21) and (C.22).

Henceforth we shall write (C.7) as

$$\begin{aligned} \dot{y} &= \beta(t)\ \underline{R}'(t)\ \underline{b}(t)\ \bar{u} + \beta(t)\ \underline{R}'(t)\ \underline{d}(t)\ \bar{v} \\ &= \beta(t)\ \underline{R}'(t)\ \underline{b}(t)\ \bar{u} + \beta(t) |\underline{R}'(t)\ d(t)|\, \mathrm{sgn}(q,y) \end{aligned} \tag{C.23}$$

and

$$\bar{J}(y(\tau),\ \tau,\ T;\ \bar{u},\ \bar{v}^*) = \frac{y^2(\tau)}{2\beta^2(\tau)} + \int_\tau^T \left[\frac{\bar{u}^2}{2} + \left| \frac{\underline{R}'(t)\ \underline{d}(t)}{\beta(t)}\ y \right| \right] dt \tag{C.24}$$

It is possible to write $\bar{J}$ as

$$\begin{aligned} \bar{J}(y(\tau),\ \tau,\ T;\ \bar{u},\ \bar{v}^*) &= \frac{y^2(\tau)}{2\beta^2(\tau)} + \phi(\tau) \left| \frac{y(\tau)}{\beta(\tau)} \right| + \frac{\phi^2(\tau)}{2} \\ &+ \frac{1}{2} \int_\tau^T (\bar{u} + \underline{R}'(t)\ \underline{b}(t)\ \phi(t)\ \mathrm{sgn}(o,y))^2\ dt \end{aligned} \tag{C.25}$$

This may be verified using the same arguments that were used to prove the identity in (C.15).

Consider the control

$$\bar{u} = -\ \underline{R}'(t)\ \underline{b}(t)\ \phi(t)\ \mathrm{sgn}(o,y) \tag{C.26}$$

If this control is substituted into (C.23), the resulting differential equation is

$$\begin{aligned} \dot{y} &= -\ \beta(t)(\underline{R}'(t)\ \underline{b}(t))^2\ \phi(t)\ \mathrm{sgn}(o,y) \\ &+ \beta(t) |\underline{R}'(t)\ \underline{d}(t)|\, \mathrm{sgn}(q,y) \end{aligned} \tag{C.27}$$

Consider now some region $\mathcal{R}$ in (y,τ) space. We shall say that $y(\tau) \in \mathcal{R}$ if a forward solution of (C.27) starting at $y(\tau)$ exists on $(\tau,T]$. This leads to the following statement.

Statement C.1.2

If $y(\tau) \in \mathcal{R}$, then the minimizing control u* is

$$u^* = - \underline{R}'(t)\, \underline{b}(t)\, \phi(t)\, \mathrm{sgn}(o,y) \tag{C.28}$$

on $(\tau,T]$ and

$$\bar{F}(y(\tau),\ \tau) = \bar{J}(y(\tau),\ \tau,\ T;\ \bar{u}^*,\ \bar{v}^*)$$

$$= \frac{y^2(\tau)}{2\beta^2(\tau)} + \phi(\tau)\left|\ \frac{y(\tau)}{\beta(\tau)}\ \right| + \frac{\phi^2(\tau)}{2} \tag{C.29}$$

These results may be proved as follows. Since the only term in (C.25) which depends upon $\bar{u}$ is the integral, and since $\bar{u}^*$ clearly minimizes this integral (i.e., the minimum value of the integral is zero), $\bar{u}^*$ must be the minimizing control. Note that $\bar{u}^* \in S^*_{\bar{u}}$, as previously assumed.

Our next effort will be to determine the region $\mathcal{R}$. Let $y^*(t,y(\tau))$ represent a forward solution of (C.27) on $[\tau,T]$ starting at $y(\tau)$. Suppose that this solution exists on $(\tau,\bar{t})$ where $\bar{t} \in (\tau,T]$. If $y^*(\bar{t},y(\tau)) \neq 0$, inspection of (C.27) indicates that $y^*(t,y(\tau))$ will exist on an interval to the right of $\bar{t}$. Suppose now that $y^*(\bar{t},y(\tau)) = 0$. For a solution to exist beyond this point there must be an interval $(\bar{t},\hat{t})$ of finite length on which either $\underline{R}'(t)\, \underline{d}(t) \equiv 0$ (implying that $y^*(t,y(\tau)) \equiv 0$ on $(\bar{t},\hat{t})$) or $|\underline{R}'(t)\, \underline{d}(t)| > \phi(t)(\underline{R}'(t)\, \underline{b}(t))^2$ (implying that $|y^*(t,y(\tau))| \neq 0$ on $(\bar{t},\hat{t})$, the sign of y^* depending on the choice of q.). This statement follows directly from an inspection of (C.27). If this statement holds for all $\bar{t} \in [\tau,T]$ where $y^*(\bar{t},y(\tau)) = 0$, then $y^*(t,y(\tau))$ will exist on $[\tau,T]$.

To determine Ω we shall work backwards in time from T. It is claimed that there must be an interval $[t_1,T]$ on which either

$$\left.\begin{array}{l} 1.\ \ \underline{R}'(t)\ \underline{d}(t) \equiv 0 \\ \text{or} \\ 2.\ \ |\underline{R}'(t)\ \underline{d}(t)| - \phi(t)(\underline{R}(t)\ \underline{b}(t))^2 > 0 \end{array}\right\} \tag{C.30}$$

This claim may be easily proved from an inspection of (3.32) and (C.6) Thus for any value of $y(\tau), \tau \in [t_1,T]$, a solution $y^*(t,y(\tau))$ will exist on $[\tau,T]$. Define the function $n(t)$ on $[t_1,T]$ as

$$n(t) \equiv 0 \tag{C.31}$$

It is obvious that $y(\tau) \in \Omega$ on $[t_1,T]$ if

$$|y(\tau)| \geq n(\tau) \tag{C.32}$$

Now consider a second interval $[t_2,t_1)$ on which neither 1 nor 2 in (C.30) hold. That is

$$\left.\begin{array}{l} |\underline{R}'(t)\ \underline{d}(t)| - \phi(t)(\underline{R}'(t)\ \underline{b}(t)) \leq 0 \\ \underline{R}'(t)\ \underline{d}(t) \neq 0 \end{array}\right\} \tag{C.33}$$

on $[t_2,t_1)$. We know from the preceding discussion that for a solution of (C.27) to exist on the entire interval (t_2,t_1), the forward solution $y^*(t,y(\tau)) \neq 0$ for $\tau \in [t_2,t_1)$ and $t \in (\tau,t_1)$. Define $n(\tau)$ for each $\tau \in [t_2,t_1)$ as the smallest value of $|y(\tau)|$ for which $y^*(t,y(\tau))$ exists on (τ,t_1). This smallest value of $|y(\tau)|$ will result in $y^*(t_1,y(\tau)) = 0$. Thus $n(\tau)$ may be determined on $[t_2,t_1)$ by solving (C.27) backwards in time from t_1 with $y(t_1) = 0$. Since $q = \pm 1$, there are two solutions. We are interested in the non-negative one. That is, on $[t_2,t_1)$, $n(\tau)$ is the non-negative backward solution of (C.27) starting at t_1 with $y(t_1) = 0$. Thus for $\tau \in [t_2,T]$, $y(\tau) \in \Omega$ if

$$|y(\tau)| \geq n(\tau) \tag{C.34}$$

Now consider a third interval $[t_3,t_2)$ on which either 1 or 2 in (C.30) holds. From previous considerations, it is known that solutions of (C.27) exist on (τ,t_2) for all $y(\tau)$, $\tau \in [t_3,t_2)$. However, for $y(\tau) \in R$, solutions must exist on $(\tau,T]$. Thus we must insure that $y(\tau)$ is such that

$$|y^*(t_2,y(\tau))| \geq n(t_2) \tag{C.35}$$

Define $n(\tau)$ on $[t_3,t_2)$ as the smallest value of $|y(\tau)|$ for which (C.35) holds. Thus on $[t_3,t_2)$, $n(\tau)$ is the non-negative backward solution of (C.27) starting at $y(t_2) = n(t_2)$. It is clear that $y(\tau) \in R$ if $|y(\tau)| \geq n(\tau)$, $\tau \in [t_3,T]$.

Continuing in this way, one can define $n(t)$ for all $t \in [t_o,T]$. We present the result as follows:

Statement C.1.3

For $t \in [t_o,T]$, $y(t) \in R$ if

$$|y(t)| \geq n(t) \tag{C.36}$$

where

$$\dot{n}(t)=\left\{\begin{array}{ll} 0 & \text{if } n(t) = 0 \text{ and either} \\ & \underline{R}'(t)\underline{d}(t) = 0 \quad \text{or} \\ & |\underline{R}'(t)\underline{d}(t)|-\phi(t)(\underline{R}'(t)\underline{b}(t))^2>0 \\ \beta(t)[|\underline{R}(t)\underline{d}(t)|-\phi(t)(\underline{R}'(t)\underline{b}(t))^2] & \text{Otherwise} \end{array}\right\} \tag{C.37}$$

and

$$n(T) = 0 \tag{C.38}$$

The differential equation for $n(t)$ follows directly from the preceding discussion.

A typical curve for $n(t)$ is shown in (y,t) space in Figure C.1. The scalar $\delta(t)$ appearing in the figure is defined in equation (3.34). The thick line represents $n(t)$ and the thin lines are $y(t)$ trajectories in $\mathcal{R}$. The dashed line in the figure is the mirror image of $n(t)$ below the t-axis, and also represents a boundary of $\mathcal{R}$.

It may be easily determined that

$$\xi(t) = \frac{n(t)}{\beta(t)} \tag{C.39}$$

where $\xi(t)$ is defined in (3.33). Using this fact and equations (C.1) to (C.6), it is clear that the claims in Theorem 3.1 follow directly from Statements C.1 to C.3.

C.2 Proof of Theorem 3.2

To prove Theorem 3.2, it must be shown that for all $y(\tau)$ in the complement of $\mathcal{R}$, a min-max solution does not exist. In Figure C.1, the interior of regions I and II represent the complement of $\mathcal{R}$. In the proof it will be assumed that $y(\tau)$ is in Region I. An analogous proof can be given for points in Region II.

Let us assume that there exists a min-max trajectory $y^*(t,y(\tau))$, starting at $y(\tau)$ which is in Region I.† We claim that

$$y^*(t_1,y(\tau)) = 0 \tag{C.40}$$

If $y^*(t,y(\tau))$ remains in Region I on (τ,t_1), then it clearly must go through the point $y(t_1) = 0$. If the trajectory leaves Region I prior to t_1, it must intersect a boundary of the region. Each boundary is a min-max trajectory which goes through the point $y(t_1) = 0$. Thus $y^*(t_1,y(\tau))$ must also go through this point, proving that (C.40) is correct.

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†It will be shown that this assumption results in a contradiction.

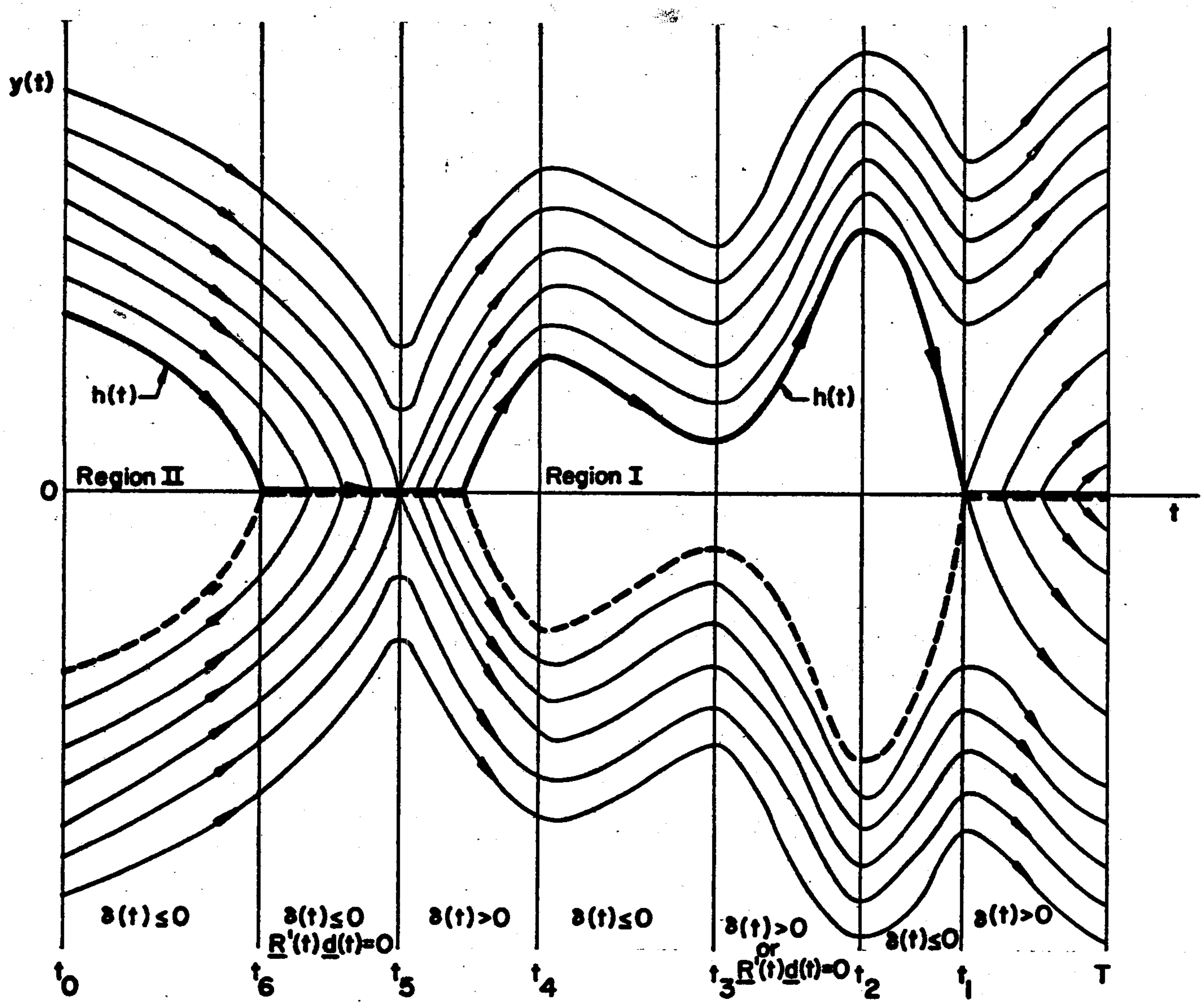

Figure C.1. Schematic representation of trajectories in (y,t) space.

Now consider the return

$$\overline{F}(y(\tau),\tau) = \min_{\overline{u} \epsilon S_{\overline{u}}} \max_{\overline{v} \epsilon S_{\overline{v}}} \left[\frac{1}{2}\int_{\tau}^{T}\left[\overline{u} - \frac{\underline{R}'(t)\underline{b}(t)}{\beta(t)} y\right]^2 dt + \frac{1}{2} y^2(T)\right] \quad (C.41)$$

It follows from the Principle of Optimality, (C.40), and (C.29) that

$$\overline{F}(y(\tau),\tau) = \frac{1}{2} \phi^2(t_1) + \min_{\overline{u} \epsilon S_{\overline{u}}} \max_{\overline{v} \epsilon S_{\overline{v}}} \left[\frac{1}{2}\int_{\tau}^{t_1}\left[\overline{u} - \frac{\underline{R}'(t)\underline{b}(t)}{\beta(t)} y\right]^2 dt\right] \quad (C.42)$$

We now claim that there must exist a τ sufficiently close to t_1 so that

$$y^*(t,y(\tau)) \neq 0 \quad (C.43)$$

on (τ,t_1). To prove this, we shall first rule out the possibility that $y^*(t,y(\tau)) \equiv 0$ on this interval, using the following argument. If $y^*(t,y(\tau)) \equiv 0$, then from (C.7)

$$\underline{R}'(t)\ \underline{b}(t)\ \overline{u}^* + R'(t)\ d(t)\ \overline{v}^* \equiv 0 \quad (C.44)$$

Since $\overline{u}^*$ cannot be a function of $\overline{v}^*$ in a min-max problem, (C.44) will hold only if $\underline{R}'(t)\ \underline{b}(t)\ u^* \equiv 0$ and $\underline{R}'(t)\ \underline{d}(t)\ \overline{v}^* \equiv 0$. Referring to equation (C-42), it is clear that

$$\min_{\overline{u} \epsilon S_{\overline{u}}} \max_{\overline{v} \epsilon S_{\overline{v}}} \int_{\tau}^{t_1} \frac{1}{2}\left[\overline{u} - \frac{\underline{R}'(t)\ \underline{b}(t)}{\beta(t)} y\right]^2 dt = 0 \quad (C.45)$$

Inspection of (C.37) indicates that $\underline{R}'(t)\ \underline{d}(t) \neq 0$ on (τ,t_1) for τ sufficiently close to t_1. Thus $\overline{v}^* \equiv 0$ on (τ,t_1). But an inspection of (C.7) clearly indicates that some other choice for $\overline{v}$, say $\overline{v} = 1$, would result in a non-zero (positive) value for the integral in (C.45). Thus $\overline{v}^* \equiv 0$ cannot be a maximizing disturbance, and $y^*(t,y(\tau)) \equiv 0$ cannot be a min-max trajectory on (τ,t_1).

Next we recall that for $\overline{u}^*$ and $\overline{v}^*$ to be admissible, they must be piecewise-continuous time functions. It follows from this observation that there must exist a τ sufficiently close to t_1 for which (C.43) holds.

For the remainder of the proof, it will be assumed that $y^*(t,y(\tau)) > 0$ on (τ,t_1). An analogous proof can be given for the case $y^*(t,y(\tau)) < 0$.

Since $y^*(t,y(\tau)) > 0$, it follows from equations (C.7), (C.40), and (C.42) that $\bar{v}^* = \text{sgn}(o, \underline{R}'(t)\ \underline{d}(t))$ on (τ,t_1). Thus (C.7) may be written as

$$\dot{y} = \beta(t)\ \underline{R}'(t)\ \underline{b}(t)\ \bar{u} + \beta(t)\ |\underline{R}'(t)\ \underline{d}(t)| \tag{C.46}$$

The control $\bar{u}^*$ may be determined by minimizing the integral

$$\frac{1}{2}\int_{\tau}^{t_1}\left[\bar{u} - \frac{\underline{R}'(t)\ \underline{b}(t)}{\beta(t)}\ y\right]^2 dt \tag{C.47}$$

subject to (C.40) and (C.46). If $\bar{u}^*$ is a minimizing control, it must satisfy the necessary conditions of the Maximum Principle [6]. To apply the principle, we first form the Hamiltonian function

$$H(y,\lambda,\bar{u}) = \frac{1}{2}\left(\bar{u} - \frac{\underline{R}'(t)\ \underline{b}(t)}{\beta(t)}\ y\right)^2$$

$$+ \lambda\ \beta(t)\ \underline{R}'(t)\ \underline{b}(t)\ \bar{u} + \lambda\ \beta(t)|\underline{R}'(t)\ \underline{d}(t)| \tag{C.48}$$

where λ is a multiplier to be determined. If $\bar{u}^*$ is a minimizing control, then it is necessary that it minimize H at each instant of time. This implies that

$$\bar{u}^* = \frac{\underline{R}'(t)\ \underline{b}(t)}{\beta(t)}\ y - \lambda(t)\ \beta(t)\ \underline{R}'(t)\ \underline{b}(t) \tag{C.49}$$

The multiplier λ satisfies

$$\dot{\lambda} = -\ H_y(y,\lambda,\bar{u}^*) = -\ (\underline{R}'(t)\ \underline{b}(t))\lambda \tag{C.50}$$

Using (C.2) it is clear that

$$\lambda(t) = \frac{C}{\beta^2(t)} \tag{C.51}$$

where C is a constant of integration depending on τ. Thus

$$\bar{u}^* = \frac{\underline{R}'(t)\ \underline{b}(t)}{\beta(t)}\ y(t) - \frac{C\ \underline{R}'(t)\ \underline{b}(t)}{\beta(t)} \qquad (C.52)$$

and

$$\dot{y} = (\underline{R}'(t)\ \underline{b}(t))^2\ y - (\underline{R}'(t)\ \underline{b}(t))^2\ C$$
$$+ \beta(t)\ |\underline{R}'(t)\ \underline{d}(t)| \qquad (C.53)$$

Since $y^*(t,y(\tau)) > 0$ on (τ,t_1) and $y^*(t_1,y(\tau)) = 0$, it follows from (C.53) that

$$(\underline{R}(t)\ \underline{b}(t))^2\ C - \beta(t)\ |\underline{R}'(t)\ \underline{d}(t)| \geq 0 \qquad (C.54)$$

must hold on (τ,t_1) for τ sufficiently close to t_1. It also follows from (C.53) that

$$y(t_1) = \beta^2(t_1)\Big[\frac{y(\tau)}{\beta^2(\tau)} - C\Big\{\int_\tau^{t_1}\Big[\frac{\underline{R}'(t)\ \underline{b}(t)}{\beta(t)}\Big]^2 dt$$
$$- \int_\tau^{t_1}\frac{|\underline{R}'(t)\ \underline{d}(t)|}{\beta(t)}\ dt\Big\}\Big] \qquad (C.55)$$

But from (C.2) and (3.32)

$$\frac{1}{\beta^2(t_1)} - \frac{1}{\beta^2(\tau)} = -\int_\tau^{t_1}\frac{(\underline{R}'(t)\ \underline{b}(t))^2}{\beta^2(t)}\ dt \qquad (C.56)$$

and

$$\frac{\phi(\tau)}{\beta(\tau)} - \frac{\phi(t_1)}{\beta(t_1)} = \int_\tau^{t_1}\frac{|\underline{R}'(t)\ \underline{d}(t)|}{\beta(t)}\ dt \qquad (C.57)$$

Using (C.37) it may also be verified that

$$n(t) = \beta(t_1)\ \phi(t_1) - \beta(t)\ \phi(t) \qquad (C.58)$$

on (τ,t_1). Noting that $y(t_1) = 0$ and using (C.55) to (C.58), it can be determined that

$$C = \beta(t_1)\,\phi(t_1) + \frac{\dfrac{\beta^2(t_1)}{\beta^2(\tau)}\,(y(\tau) - n(\tau))}{\left(\dfrac{\beta^2(t_1)}{\beta^2(\tau)} - 1\right)} \tag{C.59}$$

For $y(\tau)$ in Region I (see (C.36)), it follows from (C.2), (C.58) and (C.59) that

$$0 \leq C < \beta(t_1)\,\phi(t_1) \tag{C.60}$$

Since $\underline{R}(t)$, $\underline{b}(t)$, $\underline{d}(t)$ and $\phi(t)$ are continuous, it follows from (C.37) that

$$(\underline{R}'(t_1)\,\underline{b}(t_1))^2\,\phi(t_1) - |\underline{R}(t_1)\,\underline{d}(t_1)| = 0$$

Therefore

$$\phi(t_1) = \frac{|\underline{R}'(t_1)\,\underline{d}(t_1)|}{(\underline{R}'(t_1)\,\underline{b}(t_1))^2} \tag{C.61}$$

From (C.60) and (C.61). it is clear that

$$(\underline{R}'(t_1)\,\underline{b}(t_1))^2\,C - \beta(t_1)\,|\underline{R}'(t_1)\,\underline{d}(t_1)| < 0 \tag{C.62}$$

But since the above term is a continuous function of time,

$$(\underline{R}'(t)\,\underline{b}(t))^2\,C - \beta(t)\,|\underline{R}'(t)\,\underline{d}(t)| < 0 \tag{C.63}$$

on (τ, t_1) for τ sufficiently close to t_1. This is a contradiction of (C.54). Thus the theorem is proved.

C.3 Proof of Theorem 3.3

The proof of this theorem is based on the same type of argument used to prove Theorem 3.1. First, several simplifying transformations are made.

Let

$$z(t) = \underline{R}'(t)\,\underline{x}(t) + \theta(t) \qquad t \in [t_o, T] \tag{C.64}$$

where $\underline{R}(t)$ and $\theta(t)$ are defined in (3.44) and (3.45). Next define $\bar{v}$ as

$$\bar{v} = \left(v - \frac{1}{2}(\gamma_2(t) + \gamma_1(t))\right)\left(\frac{2}{\gamma_2(t) - \gamma_1(t)}\right) \tag{C.65}$$

on $[t_o, T]$. Let $S_{\bar{v}}$ be the set of all piecewise-continuous time functions on $[t_o, T]$ such that if $\bar{v} \in S_{\bar{v}}$, then $|\bar{v}| \leq 1$. Note that $v \in S_v \Leftrightarrow \bar{v} \in S_{\bar{v}}$. Define the vector

$$\underline{d}(t) = \frac{1}{2}(\gamma_2(t) - \gamma_1(t))\,\underline{c}(t) \tag{C.66}$$

on $[t_o, T]$. Finally, define the functional

$$\bar{J}(z(t_o), t_o, T; u, \bar{v}) = J(\underline{x}(t_o), t_o, T; u, v) \tag{C.67}$$

From (3.42) and (3.43), it is clear that

$$\bar{J}(z(t_o), t_o, T; u, \bar{v}) = \frac{1}{2} z^2(T) \tag{C.68}$$

It is easily verified that z(t) satisfies

$$\dot{z} = \underline{R}'(t)\,\underline{b}(t)\,u + \underline{R}'(t)\,\underline{d}(t)\,\bar{v} \tag{C.69}$$

Our objective is to determine the return function $\bar{F}(y(\tau),\tau)$ which satisfies

$$\bar{F}(y(\tau),\tau) = \min_{u \in S_u} \max_{\bar{v} \in S_{\bar{v}}} \bar{J}(z(\tau), \tau, T; u, \bar{v}) \tag{C.70}$$

To do this, we shall work backwards from time T. Suppose that there is an interval $[t_1, T]$ on which $|\underline{R}(t)'\,\underline{d}(t)| - |\underline{R}(t)\,\underline{b}(t)| > 0$.† Let t_1 be the smallest time on $[t_o, T]$ for which this is true. Inspection of (C.68) to (C.70) clearly indicates that

†An analogous proof may be given for the case $|\underline{R}'(t)\,\underline{d}(t)| - |\underline{R}'(t)\,\underline{b}(t)| < 0$ on $[t_1, T]$.

$$\bar{v}^* = \mathrm{sgn}(o,\underline{R}'(t)\ \underline{d}(t))\ \mathrm{sgn}(q,z(t));\ q = 1 \text{ or } -1 \tag{C.71}$$

and

$$u^* = -\ \mathrm{sgn}(o,\underline{R}'(t)\ \underline{b}(t)\ z(t)) \tag{C.72}$$

on this interval. Note that the resulting min-max trajectory $z^*(t,z(\tau))$, $\tau \in [t_1,T]$, does not change sign on (τ,T). Since

$$z(T) = z(\tau) + \int_\tau^T [|\underline{R}'(t)\ \underline{d}(t)|\mathrm{sgn}(q,z^*) - |\underline{R}(t)\ \underline{b}(t)|\mathrm{sgn}(o,z^*)]\ dt \tag{C.73}$$

it is readily established that

$$\bar{F}(z(\tau),\tau) = \frac{1}{2}\ z^2(\tau) + \phi(\tau)|z(\tau)| + \frac{1}{2}\ \phi^2(\tau) \tag{C.74}$$

where $\phi(\tau)$ is defined in (3.46). An inspection of equation (3.47) indicates that $\xi(t) \equiv 0$ on $[t_1,T]$. Thus we have shown that if $|z(\tau)| \geq \xi(\tau)$, $\tau \in [t_1,T]$, then u^*, v^* and $\bar{F}$ are given by (C.71), (C.72) and (C.74).

Now consider a second interval $[t_2,t_1)$ on which $|\underline{R}'(t)\ \underline{b}(t)| - |\underline{R}'(t)\ \underline{d}(t)| \geq 0$. It follows from (C.74) and the Principle of Optimality that

$$\bar{F}(z(\tau),\tau) = \min_{u \in S_u}\ \max_{\bar{v} \in S_{\bar{v}}} \left[\frac{1}{2}\ z^2(t_1) + \phi(t_1)|z(t_1)| + \frac{1}{2}\ \phi^2(t_1)\right] \tag{C.75}$$

for $\tau \in [t_2,t_1)$. If $|z(\tau)|$ is sufficiently large, it is clear from (C.69) that for any admissable u and $\bar{v}$, the resulting trajectory $z(t,z(\tau))$ will not change sign on (τ,t_1). Under such circumstances, inspection of (C.69) and (C.75) clearly indicates that

$$\left.\begin{aligned} &\bar{v}^* = \mathrm{sgn}(o,\underline{R}'(t)\ \underline{d}(t))\ \mathrm{sgn}(q,z) \\ &\text{and} \\ &u^* = -\ \mathrm{sgn}(o,\underline{R}'(t)\ \underline{b}(t))\ \mathrm{sgn}(o,z) \end{aligned}\right\} \tag{C.76}$$

Using (C.73), which now holds on $(\tau,T]$, it is clear that

$$\bar{F}(z(\tau),\tau) = \frac{1}{2} z^2(\tau) + \phi(\tau)|z(\tau)| + \frac{1}{2} \phi^2(\tau) \qquad (C.77)$$

Of course, we must insure that $|z(\tau)|$ is large enough so that $z^*(t,z(\tau))$ does not change sign of (τ,t_1). It is clear that the smallest value of $|z(\tau)|$ for which this is so is the one which will result in $z^*(t_1,z(\tau))=0$. Thus by computing the non-negative, backward solution of (C.69) using u^* and $\bar{v}^*$ as given in (C.76), and starting at $z(t_1) = 0$, this smallest value for $|z(\tau)|$ can be found. It may easily be seen from (3.47) that $\xi(t)$ is this backward solution on $(\tau,t_1]$.

This argument may be continued on successive intervals where $|\underline{R}'(t)\, \underline{d}(t)| - |\underline{R}'(t)\, \underline{b}(t)|$ is either positive or non-positive until the initial time t_o is reached. The end result is summarized as follows. If for $\tau \epsilon [t_o,T]$, $|z(\tau)| \geq \xi(\tau)$, then u^*, $\bar{v}^*$ and $\bar{F}(z(\tau),\tau)$ are given by equations (C.76) and (C.77). From this result, and equations (C.64) to (C.66), it is clear that the theorem has been proved.

C.4 <u>Proof of Theorem 3.4</u>

To prove this theorem, it is convenient to refer to Figure C.1. If $n(t)$ and $y(t)$ are replaced by $\xi(t)$ and $z(t)$, the figure can be used to represent (z,t) space.† For values of $z(t)$ lying outside of or on the thick solid and dashed lines, u^* and $\bar{v}^*$ and $\bar{F}(z(t),t)$ are given by Theorem 3.3. The light lines in the figure represent min-max trajectories.

†In (z,t) space the boundary for Region II would be determined by the two trajectories arriving at $y(t_5) = 0$ rather than the two arriving at $y(t_6) = 0$ as shown. This follows from equation (3.47).

We are now primarily concerned with points which lie in Regions I and II. Let us assume that $z(\tau)$ is in Region I. A similar proof of the theorem can be given for points in Region II.

It is claimed that for any admissible u and $\bar{v}$, the resulting trajectory $z(t,z(\tau))$ must go through the point

$$z(t_1,z(\tau)) = 0 \qquad (C.78)$$

provided that u^* and $\bar{v}^*$ given in (C.76) are used if the trajectory leaves Region I. The reasoning here is the same as that used in the proof of Theorem 3.2. If $z(t,z(\tau))$ remains in Region I on (τ,t_1), then it clearly must go through the point $z(t_1) = 0$. If the trajectory leaves Region I prior to time t_1, it must intersect a boundary. Since each boundary is a min-max trajectory going through $z(t_1) = 0$, the expression in (C.78) must hold.

We may use the Principle of Optimality to write

$$\bar{F}(z(\tau),\tau) = \min_{u \in S_u} \max_{\bar{v} \in S_{\bar{v}}} \left[\frac{1}{2} z^2(t_1) + \phi(t_1)|z(t_1)| + \frac{1}{2}\phi^2(t_1)\right] \qquad (C.79)$$

Using (C.78), it is clear that

$$\bar{F}(z(\tau),\tau) = \frac{1}{2}\phi^2(t_1)$$

Note that t_1 is the first zero of $\xi(t)$ on the interval $[\tau,T]$. This proves the theorem.